REGULAR DEL PEZZO SURFACES WITH IRREGULARITY

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ABSTRACT. We construct the first examples of regular del Pezzo surfaces X for which $h^1(X, \mathcal{O}_X) > 0$. We also find a restriction on the integer pairs that are possible as the anti-canonical degree K_X^2 and irregularity $h^1(X, \mathcal{O}_X)$ of such a surface.

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INTRODUCTION

0.1. **Regular varieties.** Any variety defined over a finitely generated extension k of a perfect (e.g. algebraically closed) field $\mathbb{F}$ can be viewed as the generic fibre of a morphism of $\mathbb{F}$-varieties $\mathcal{X} \to \mathcal{B}$ such that k is the function field of the base $\mathcal{B}$. In this way, the geometry of varieties over imperfect fields is relevant to the understanding of the birational geometry of varieties over algebraically closed fields of positive characteristic. One main difficulty that arises is that, unlike over perfect fields, the notions of smoothness and regularity diverge: a smooth variety is necessarily regular, but a regular variety may not be smooth.

Definition 0.1.1. A variety X is defined to be *regular* provided that the local coordinate ring $\mathcal{O}_{X,x}$ is a regular local ring at all points $x \in X$. A k-variety X is *smooth* over k provided that it is geometrically regular (recalling that a k-variety X is said to satisfy a property *geometrically* if the base change $X_{\bar{k}}$ to the algebraic closure satisfies the given property) .

The notion of smoothness is well-behaved, due largely to the fact that a k-variety X is smooth if and only if the cotangent sheaf $\Omega_{X/k}$ is a vector bundle of rank equal to the dimension of X. Regularity, like smoothness, is a local property, and can be described in terms of the latter as follows: a k-variety X is regular if and only if there exists a smooth $\mathbb{F}$-variety $\mathcal{X}$ and a morphism $\mathcal{X} \to \mathcal{B}$ of which X is the generic fibre. In characteristic 0, a general fibre of a morphism between smooth varieties is smooth, yet in positive characteristic it is common for such morphisms to admit no smooth fibres. In fact, the collection of generic fibres of morphisms between smooth $\mathbb{F}$-varieties that admit no smooth fibres precisely comprises the regular, non-smooth varieties over finitely generated

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field extensions of $\mathbb{F}$. A standard example is the generic fibre of the family $(y^3 = x^2 + t) \subseteq \mathbb{A}^2 \times \mathbb{A}^1$ of cuspidal plane curves, parameterized by the affine coordinate t over a field of characteristic 2.

0.2. **New results.** Our study focuses on regular del Pezzo surfaces, a class of varieties that, as we discuss in §0.3, arises naturally in the context of the minimal model program.

Definition 0.2.1. A *del Pezzo scheme* over a field k is defined to be a 2-dimensional, projective, Gorenstein scheme X of finite-type over $k = H^0(X, \mathcal{O}_X)$ which is *Fano*, that is, for which the inverse of the dualizing sheaf, ω_X^{-1}, is an ample line bundle. A *del Pezzo surface* is a del Pezzo scheme that is an integral scheme.

This paper answers affirmatively the question of whether there exist regular del Pezzo surfaces X that are geometrically non-normal or geometrically non-reduced by constructing examples of each type which have positive irregularity $h^1(X, \mathcal{O}_X) = 1$. We also find a characteristic-dependent restriction on the anti-canonical degree of regular del Pezzo surfaces that have a given positive irregularity $q := h^1(X, \mathcal{O}_X) > 0$. The main result (represented graphically in Figure 1) can be concisely summarized as follows:

Main Theorem.

(1) *There exist regular del Pezzo surfaces, X_1 and X_2, with irregularity $h^1(X_i, \mathcal{O}_{X_i}) = 1$ and of degrees $K_{X_1}^2 = 1$ and $K_{X_2}^2 = 2$. The surface X_1 is geometrically integral and defined over the field $\mathbb{F}_2(\alpha_0, \alpha_1, \alpha_2)$ while X_2 is geometrically non-reduced and defined over the index-2 subfield $\mathbb{F}_2(\alpha_i\alpha_j : 0 \leq i, j \leq 3) \subseteq \mathbb{F}_2(\alpha_0, \alpha_1, \alpha_2, \alpha_3)$.*

(2) *If X is a normal, local complete intersection (l.c.i.) del Pezzo surface (e.g. a regular del Pezzo surface) with irregularity $q > 0$ and anti-canonical degree $d = K_X^2$ over a field of characteristic p, then*

$$q \geq \frac{d(p^2 - 1)}{6}. \tag{0.2.2}$$

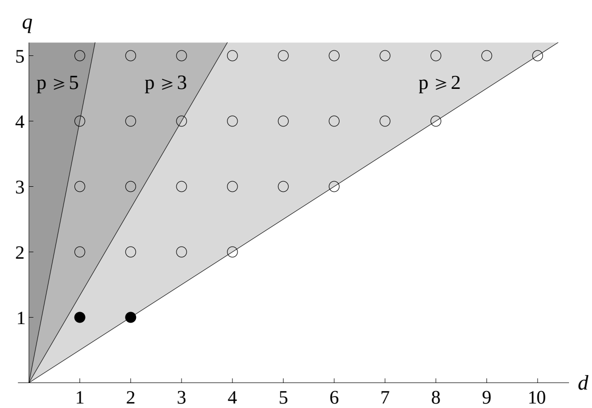

FIGURE 1. Circles represent possible values for the degree d and irregularity q of an l.c.i. and normal del Pezzo surface with positive irregularity $q > 0$. Solid dots represent actual values attained by the regular del Pezzo surfaces constructed in §3.2 and §3.3. Shaded regions demonstrate how the inequality (0.2.2) becomes more restrictive as the characteristic grows.

As our proof of (1) is constructive, it should be possible to obtain concrete descriptions of the geometry in each example. We do so for the degree one surface X_1, proving by explicit computation in Proposition 4.0.1 a detailed version of the following proposition.

Proposition. *There exists a regular form, Z, of a double plane in $\mathbb{P}^3$ and a finite, inseparable morphism $f : Z \to X_1$ of degree $p = 2$. Moreover, if $\bar{Z}$ and $\bar{X}_1$ denote the geometric base changes of Z and X_1, respectively, then this construction has the following properties:*

(1) The induced morphism $f^{red} : \mathbb{P}^2 \cong \bar{Z}^{red} \to \bar{X}_1$ is the normalization of $\bar{X}_1$.
(2) The singular locus of $\bar{X}_1$ is a rational cuspidal curve C of arithmetic genus one.
(3) The inverse image of C under f^{red} is a non-reduced double line in $\mathbb{P}^2$.

0.3. **Motivation from the minimal model program.** Among the varieties over function fields, Fano varieties such as del Pezzo surfaces are of particular interest, due to their prominent role in the minimal model program. In brief, the goal of the program is to understand the birational geometry of a variety X by constructing a birational model $\hat{X}$ whose canonical divisor $K_{\hat{X}}$ is a nef divisor; one calls such a variety $\hat{X}$ a *minimal model* of X. If $\hat{X}$ is smooth, then the terminology is justified: $\hat{X}$ is minimal in the sense that any birational morphism $\hat{X} \to X'$ to a smooth variety X' is an isomorphism (cf. [1, Prop. 1.45]).

If X is not itself a minimal model, then there exist effective curves $C \subseteq X$ that pair negatively with the canonical divisor, $C.K_X < 0$. The strategy for constructing $\hat{X}$ is to attempt to contract precisely these negative curves via birational morphisms $f : X \to Y$ and then to partially resolve any serious singularities that were introduced. However, the contraction morphism associated to a certain negative curve may not be birational, and the contracted variety Y may be of lower dimension, as is the case, say, for ruled surfaces. Since the curves contained in fibres of f each pair negatively with K_X, the fibres of f are therefore Fano schemes by Kleiman's criterion for ampleness. In other words, the contraction morphism $f : X \to Y$ realizes X as a Fano fibre-space.

When X is a smooth 3-fold over an algebraically closed field, theorems of Mori [13] and Kollár [9] guarantee that any given extremal ray in the cone of effective curves pairing negatively with K_X can be contracted by a morphism $f : X \to Y$ to a normal variety Y. Furthermore, they classify these contraction morphisms: either f is birational, equal to the inverse of the blowing-up of a point or a smooth curve in Y, or $f : X \to Y$ is a Fano fibration over a smooth variety Y of dimension at most 2. If Y is a point, then X is itself a Fano 3-fold, while if Y is a surface, then X is a conic bundle over Y.

Our case of interest is when Y is a curve, as then $f : X \to Y$ is a del Pezzo fibration. Since X is smooth, the generic fibre of the fibration is a regular del Pezzo surface over the function field of Y. In characteristic 0, regular del Pezzo surfaces are smooth, and there are some results toward a birational classification of these del Pezzo fibrations (cf. [5] for a recent survey). In positive characteristic, however, the generic del Pezzo surface is potentially non-smooth, and the situation is not so clear. Indeed, Kollár asks whether these regular del Pezzo surfaces can be geometrically non-normal, or even geometrically non-reduced, but remarks that understanding this phenomenon seems complicated, especially in characteristic 2 (cf. [9, Rem. 1.2]).

0.4. **Regular forms and the classification of del Pezzo surfaces.** We can also contextualize our results in terms of the classification of del Pezzo surfaces over an algebraically closed field. In particular, we will see how our Main Theorem makes progress toward determining which singular (possibly non-normal or non-reduced) del Pezzo schemes over algebraically closed fields admit regular k-forms for some subfield k.

Definition 0.4.1. Let K/k be an extension of fields. Given a K-variety $\bar{X}$, one says that a k-variety X is a *(k-)form* of $\bar{X}$ provided that there exists an isomorphism $\bar{X} \cong X \times_k K$.

We recall the classification of del Pezzo surfaces X over an algebraically closed field. When X is normal, Hidaka and Watanabe [7] prove that either X is a rational surface with singularities at worst rational double points or X is a cone over an elliptic curve. Not all of these surfaces admit regular forms, as Hirokado [8] and Schröer [17] show how the existence of a regular form puts restrictions on the possible singularities.

In the course of proving the classification result, Hidaka and Watanabe [7] prove that all normal del Pezzo surfaces over an algebraically closed field satisfy $H^1(X, \mathcal{O}_X) = 0$. Over the complex numbers, this cohomological vanishing $H^1(X, \mathcal{O}_X) = 0$ can be viewed as a consequence of the Kodaira vanishing theorem for normal surfaces (cf. [14]), since $H^1(X, \mathcal{O}_X)$ is Serre-dual to the group $H^1(X, \omega_X)$ and ω_X is the inverse of an ample line bundle.

Reid [15] classifies the non-normal del Pezzo surfaces. He shows that such surfaces X are formed from rational, normal varieties X^ν by collapsing a (possibly non-smooth) conic to a rational curve C that is either smooth or has wildly cuspidal singularities (i.e. cuspidal singular points of order divisible by the prime characteristic $p > 0$). We remark that for these surfaces, the irregularity is equal to the arithmetic genus of the curve of singularities C, that is, $h^1(X, \mathcal{O}_X) = h^1(C, \mathcal{O}_C)$. In particular, when C is smooth, X is a non-normal del Pezzo surface with $H^1(X, \mathcal{O}_X) = 0$.

When C is wildly cuspidal, Reid shows that the normalization X^ν is the cone over a rational curve of degree $d \geq 1$ and the normalization morphism $\phi : X^\nu \to X$ is the contraction to C of the non-reduced double structure D on a ruling D^{red} in X^ν. Moreover, the restriction of ϕ to the ruling D^{red} gives a desingularization of C. This construction requires the cusps of C to be wild, because otherwise the resulting variety X is not Gorenstein (cf. [15, §4.4]). Such examples X are non-normal del Pezzo surfaces of anti-canonical degree $K_X^2 = d$ and irregularity $h^1(X, \mathcal{O}_X) = h^1(C, \mathcal{O}_C) > 0$. Reid constructs explicit surfaces X where the curve C has cusps of arbitrarily large order, showing that the irregularity of a non-normal del Pezzo surface may be arbitrarily large. Such surfaces are arguably the most pathological examples of del Pezzo surfaces.

In light of this classification, a scheme $\bar{X}$ admits a k-form that is a del Pezzo surface over k with irregularity $h^1(X, \mathcal{O}_X) > 0$ only if $\bar{X}$ is a non-normal del Pezzo surface or $\bar{X}$ is a non-reduced del Pezzo scheme. Main Theorem (1) asserts that regular forms can exist in either case, and Main Theorem (2) provides a numerical inequality that, in particular, rules out a large class of non-normal del Pezzo surfaces that could potentially admit regular forms.

0.5. **A prior example.** Acknowledging Reid's non-normal classification, Kollár remarks in [11, Rem. 5.7.1] on the possibility that regular del Pezzo surfaces could have positive irregularity. He ultimately leaves the issue unresolved, although his question is repeated later by Schröer in [16]. There Schröer constructs an interesting example of a normal del Pezzo surface Y in characteristic 2 that is a local complete intersection (l.c.i.) and regular away from one singular point y_∞, and has irregularity $h^1(Y, \mathcal{O}_Y) = 1$. This variety Y is a form of the example of Reid whose normalization morphism is described as the collapse of a non-reduced double line in $\mathbb{P}^2$ to a reduced cuspidal curve C with arithmetic genus 1. Schröer's method of construction is to begin with any imperfect field k of characteristic 2 along with a non-normal k-form X of the variety constructed by Reid. Schröer then studies actions of the infinitesimal group scheme α_2 on X. He uses one such action to twist the field of definition, thus obtaining the twisted form Y which he proves to be l.c.i. and normal. Schröer shows moreover that no α_2-twisting of the variety X can remove the singularity at y_∞, and hence his surface Y is an optimal one obtainable by this method.

0.6. **A brief outline.** The numerical bound in Main Theorem (2) is obtained in §1 by studying the inseparable degree p covers associated to Frobenius-killed classes in the first cohomology group of pluri-canonical line bundles on X. Such covers were studied by Ekedahl in [3] and shown to have peculiar properties, which we interpret to deduce the inequality (0.2.2). The notion of algebraic foliation on a regular (possibly non-smooth) variety is developed in §2, where we extend results of Ekedahl [2] from the smooth case. The surfaces X_1 and X_2 mentioned in Main Theorem (1) are exhibited as quotients by explicit algebraic foliations on a regular form of a non-reduced double plane in projective 3-space in §3. We conclude in §4 with a detailed study of the example X_1, a regular and geometrically integral del Pezzo surface with $h^1(X_1, \mathcal{O}_{X_1}) = 1$.

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NOTATION

- All fields are assumed to be of characteristic $p \geq 2$.
- A *variety* over a field k is a finite-type, integral k-scheme.
- $k(X)$ denotes the function field of a k-variety X.
- $k_X := H^0(X, \mathcal{O}_X)$ denotes the field of global functions of a proper k-variety X.
- K_X denotes the canonical divisor associated to the dualizing sheaf ω_X of a Gorenstein variety X.
- $d = K_X^2$ denotes the anti-canonical degree of a del Pezzo surface X, computed as the self-intersection number over the field $k_X = H^0(X, \mathcal{O}_X)$.
- $h^i(X, \mathscr{F}) := \dim_{k_X} H^i(X, \mathscr{F})$ denotes the dimension over the field $k_X = H^0(X, \mathcal{O}_X)$ of the ith cohomology group of a sheaf $\mathscr{F}$ on a proper variety X.
- $q = h^1(X, \mathcal{O}_X)$ denotes the irregularity of a proper surface X.
- $\chi(\mathscr{F}) := \sum_i (-1)^i h^i(X, \mathscr{F})$ denotes the Euler characteristic of the coherent sheaf $\mathscr{F}$ on a proper variety X.
- $\mathbf{F}_X : X \to X$ denotes the absolute Frobenius morphism of a scheme X.
- $\mathbf{F}_{X/S}$ denotes the Frobenius morphism relative to a morphism of schemes $X \to S$.
- $\Omega_{Z/S}$ denotes the sheaf of relative Kähler differentials of an S-scheme Z.
- $T_{Z/S} := \mathscr{H}om(\Omega_{Z/S}, \mathcal{O}_Z)$ denotes the relative tangent bundle of an S-scheme Z.

1. NUMERICAL BOUNDS ON DEL PEZZO SURFACES WITH IRREGULARITY

The goal of this section is to find a restriction on the possible integer pairs (d, q) that exist as the degree $d = K_X^2$ and irregularity $q = h^1(X, \mathcal{O}_X)$ of a normal, l.c.i. del Pezzo surface X over a field k, under the assumption that $q \neq 0$. Our method is to study the torsors, for certain non-reduced group schemes $\alpha_{\mathscr{L}}$, associated to Frobenius-killed classes in the first cohomology group of pluri-canonical line bundles $\mathscr{L} := \omega_X^{\otimes m}$ on X. Originally studied by Ekedahl in [2, 3], the existence of such torsors are often used as a technique to work around the failing of Kodaira vanishing in characteristic $p > 0$.

1.1. **$\alpha_{\mathscr{L}}$-torsors.** We briefly summarize here the basic properties of $\alpha_{\mathscr{L}}$-torsors, but we refer the reader to [3] or [10, §II.6.1] for more detailed accounts.

Let $\mathscr{L}$ be a line bundle on a variety X over a field k of characteristic p such that $H^1(X, \mathscr{L}) \neq 0$. We note that if $\mathscr{L}$ is the inverse of an ample line bundle, then this would be an example of the

Kodaira non-vanishing phenomenon. Assume as well that pulling-back by the absolute Frobenius morphism $\mathbf{F}_X : X \to X$ does not yield an injective homomorphism from $H^1(X, \mathscr{L})$, that is, there exists a nonzero class $\bar{\xi} \in H^1(X, \mathscr{L})$ for which

$$\mathbf{F}_X^*(\bar{\xi}) = 0 \in H^1(X, \mathscr{L}^{\otimes p}).$$

The Frobenius pull-back defines a surjective homomorphism of group schemes over X from $\mathscr{L}$ to $\mathscr{L}^{\otimes p}$. Let $\alpha_{\mathscr{L}}$ be the group scheme defined as the kernel of this homomorphism, which by definition sits in the short exact sequence of group schemes,

$$0 \to \alpha_{\mathscr{L}} \to \mathscr{L} \xrightarrow{\mathbf{F}_X^*} \mathscr{L}^{\otimes p} \to 0. \tag{1.1.1}$$

Locally the group scheme $\alpha_{\mathscr{L}}$ is isomorphic to the constant non-reduced group scheme α_p, whose fibre over X is the kernel of the pth power endomorphism of the additive group $\mathbb{G}_a$.

The long exact sequence in cohomology associated to (1.1.1) shows that the class $\bar{\xi}$ comes from a nonzero class $\xi \in H^1(X, \alpha_{\mathscr{L}})$ that is determined up to an element of the cokernel of $\mathbf{F}_X^* : H^0(X, \mathscr{L}) \to H^0(X, \mathscr{L}^{\otimes p})$. Via Čech cohomology, one sees that ξ gives rise to a nontrivial $\alpha_{\mathscr{L}}$-torsor $f : Z \to X$. The morphism $f : Z \to X$ is purely inseparable of degree p because $\alpha_{\mathscr{L}}$ is a non-reduced finite group scheme of degree p over X.

To describe this $\alpha_{\mathscr{L}}$-torsor more explicitly, notice that a Frobenius-killed class $\bar{\xi} \in H^1(X, \mathscr{L})$ corresponds to a non-split extension of vector bundles,

$$0 \to \mathcal{O}_X \xrightarrow{i} \mathscr{E} \xrightarrow{\pi} \mathscr{L}^{-1} \to 0,$$

for which there is some splitting $\sigma : \mathscr{L}^{\otimes -p} \to \mathbf{F}_X^* \mathscr{E}$ of the morphism $F_X^* \pi$. We note that the choice of splitting is determined up to an element of $H^0(X, \mathscr{L}^{\otimes p})$. The affine algebra $f_* \mathcal{O}_Z$ is the quotient of the symmetric algebra $\mathrm{Sym}^*(\mathscr{E})$ by the ideal generated by $1 - i(1)$ as well as the image of σ in $\mathbf{F}_X^* \mathscr{E} \subseteq \mathrm{Sym}^p(\mathscr{E}) \subseteq \mathrm{Sym}^*(\mathscr{E})$. Two splittings yield isomorphic quotients precisely when they differ by an element in the image of $\mathbf{F}_X^* : H^0(X, \mathscr{L}) \to H^0(X, \mathscr{L}^{\otimes p})$. Thus we see that this explicit construction is also determined by the data of some class $\xi \in H^1(X, \alpha_{\mathscr{L}})$ lifting $\bar{\xi} \in H^1(X, \mathscr{L})$.

Proposition 1.1.2 (Ekedahl). *If X is a normal, projective, Gorenstein (resp. l.c.i.) variety and $f : Z \to X$ a nontrivial $\alpha_{\mathscr{L}}$-torsor for some line bundle $\mathscr{L}$, then Z is a projective, Gorenstein (resp. l.c.i.) variety satisfying:*

(1) $\omega_Z \cong f^*(\omega_X \otimes \mathscr{L}^{\otimes p-1})$,
(2) $\chi(f_* \mathcal{O}_Z) = \sum_{i=0}^{p-1} \chi(\mathscr{L}^{\otimes -i})$.

Proof. Showing that Z is integral with $\omega_Z \cong f^*(\omega_X \otimes \mathscr{L}^{\otimes p-1})$ when X is normal can be found in [3, §1] or [10, Prop. II.6.1.7]. From the explicit description of Z given above, we obtain a filtration of $f_* \mathcal{O}_Z$, given by the images of $\mathrm{Sym}^i(\mathscr{E})$, whose successive quotients are isomorphic to $\mathscr{L}^{\otimes -i}$, for $0 \leq i < p$ (cf. [3, Prop. 1.7]); this immediately yields the Euler characteristic formula in (2). Finally, if X is l.c.i., then Z is too as it embeds in the affine bundle $\mathbf{Spec}\ \mathrm{Sym}^*(\mathscr{E})/(1 - i(1))$ over X as the Cartier divisor defined locally by $\sigma(s)$, where s is a local generator of $\mathscr{L}^{\otimes -p}$. ∎

We intend to use Proposition 1.1.2 (2) to relate the Euler characteristic of the structure sheaf of a normal, l.c.i. del Pezzo surface X to that of a nontrivial $\alpha_{\mathscr{L}}$-torsor $f : Z \to X$. Yet, if the fields $k_Z := H^0(Z, \mathcal{O}_Z)$ and $k_X := H^0(X, \mathcal{O}_X)$ do not coincide, then the Euler characteristics $\chi(\mathcal{O}_Z)$ and $\chi(f_* \mathcal{O}_Z)$ differ by a factor of $[k_Z : k_X]$:

$$\chi(f_* \mathcal{O}_Z) = [k_Z : k_X] \cdot \chi(\mathcal{O}_Z).$$

The following easy lemma controls this factor, showing it is either 1 or p.

Lemma 1.1.3. *If $f : Z \to X$ is a finite dominant morphism of degree d from a proper variety Z to a normal, proper variety X over k, then $k_Z := H^0(Z, \mathcal{O}_Z)$ is a field extension of $k_X := H^0(X, \mathcal{O}_X)$ whose degree divides d, that is,*

$$[k_Z : k_X] \mid d.$$

Proof. There are field extensions $k_X \subseteq k(X) \subseteq k(Z)$ and $k_Z \subseteq k(Z)$. Because X is normal and f is finite, $k_Z \cap k(X) = k_X$. Therefore, $k_Z \otimes_{k_X} k(X)$ is a subfield of $k(Z)$, of degree $[k_Z : k_X]$ over $k(X)$, and hence $[k_Z : k_X]$ divides $[k(Z) : k(X)] = d$. ∎

1.2. **Normal del Pezzo surfaces of local complete intersection.** Let X be a normal, l.c.i. del Pezzo surface over a field k such that for some integer n the cohomology group $H^1(X, nK_X) \neq 0$ (e.g. X is a regular del Pezzo surface with irregularity and $n = 1$). We will see that the construction of the previous subsection can be used to create a degree p inseparable morphism $f : Z \to X$ whose existence puts restrictions on the possible pairs of integers (d, q) that arise as the degree d and irregularity q of such X. The normalcy condition is used to ensure the integrality of Z, and the l.c.i. condition guarantees that we may use the following version of the Riemann-Roch theorem:

Theorem 1.2.1 (Riemann-Roch)**.** *If D be a Cartier divisor on a 2-dimensional variety X of local complete intersection, then*

$$\chi(\mathcal{O}_X(D)) = \chi(\mathcal{O}_X) + \frac{1}{2} D.(D - K_X).$$

Proof. The Grothendieck-Riemann-Roch theorem asserts for any line bundle $\mathscr{L}$ on X,

$$\chi(\mathscr{L}) = \int_X \mathrm{ch}(\mathscr{L}) \frown (\mathrm{td}(T_{vir}) \frown [X]), \tag{1.2.2}$$

where T_{vir} is the virtual tangent bundle of X (cf. [4, Cor. 18.3.1(b)]). The Todd class is given by $\mathrm{td}(T_{vir}) = 1 + \frac{1}{2} c_1(T_{vir}) + \frac{1}{12}(c_1(T_{vir})^2 + c_2(T_{vir}))$, and the Chern character by $\mathrm{ch}(\mathscr{L}) = 1 + c_1(\mathscr{L}) + \frac{1}{2} c_1(\mathscr{L})^2$. Taking $\mathscr{L} := \mathcal{O}_X$, we see that $\chi(\mathcal{O}_X) = \frac{1}{12} \int_X (c_1(T_{vir})^2 + c_2(T_{vir})) \frown [X]$. Substituting these expressions into (1.2.2) for $\mathscr{L} := \mathcal{O}_X(D)$ results in the formula

$$\chi(\mathcal{O}_X(D)) = \chi(\mathcal{O}_X) + \frac{1}{2} \int_X D.(D + c_1(T_{vir})).$$

We finish by noting that $c_1(T_{vir}) = -K_X$, due to the adjunction formula for local complete intersections. ∎

The main result of this section is the following:

Theorem 1.2.3. *Let X be a normal, l.c.i. del Pezzo surface with irregularity $q_X = h^1(X, \mathcal{O}_X)$.*

(1) If $q_X > 0$ then there exists a positive integer m such that the line bundle $\mathscr{L} := \omega_X^{\otimes m}$ has the following property:

$(*)$ the absolute Frobenius pullback $\mathbf{F}_X^* : H^1(X, \mathscr{L}) \to H^1(X, \mathscr{L}^{\otimes p})$ has a nontrivial kernel.

(2) If $\mathscr{L}$ is a line bundle that satisfies $(*)$ and is numerically equivalent to $\omega_X^{\otimes m}$ for some integer m, then there exists a nontrivial $\alpha_{\mathscr{L}}$-torsor Z that is an l.c.i. del Pezzo surface of anti-canonical degree

$$K_Z^2 = p^{1-e}(1 + m(p-1))^2 K_X^2. \tag{1.2.4}$$

The field $k_Z := H^0(Z, \mathcal{O}_Z)$ is an extension of $k_X := H^0(X, \mathcal{O}_X)$ of degree p^e with $e \in \{0,1\}$, and if $q_Z := h^1(Z, \mathcal{O}_Z)$ denotes the irregularity of Z, then

$$p^e(1-q_Z) = p - pq_X + \frac{mp(p-1)K_X^2}{12}\left(3 + m(2p-1)\right). \tag{1.2.5}$$

Proof. The existence of an integer m as in (1) is an immediate consequence of Serre's theorems on duality and vanishing of higher cohomology. Let $\mathscr{L}$ be any line bundle satisfying the hypothesis of (2), and let Z be any $\alpha_{\mathscr{L}}$-torsor Z associated to a nonzero Frobenius-killed cohomology class $\bar{\xi} \in H^1(X, \mathscr{L})$. By Proposition 1.1.2, the torsor Z is an l.c.i. variety with dualizing sheaf $\omega_Z \cong f^*(\omega_X \otimes \mathscr{L}^{\otimes p-1})$. Hence, we can compute the anti-canonical degree of Z (over k_Z) as

$$K_Z^2 = \frac{\deg f}{[k_Z : k_X]} \cdot (1 + m(p-1))^2 K_X^2.$$

Since $\deg f = p$, Lemma 1.1.3 implies that $[k_Z : k_X] = p^e$ with $e \in \{0,1\}$, which proves (1.2.4).

Since both ω_X^{-1} and $\mathscr{L}^{-1}$ are ample line bundles on X and f is a finite morphism, the line bundle ω_Z^{-1} is ample and Z is therefore an l.c.i. del Pezzo surface. Moreover, Proposition 1.1.2 gives the equality $\chi(f_*\mathcal{O}_Z) = \sum_{i=0}^{p-1} \chi(\mathscr{L}^{\otimes -i})$. The Riemann-Roch theorem shows that $\chi(\mathscr{L}^{\otimes -i})$ is independent of the numerical equivalence class of $\mathscr{L}^{\otimes -i}$ and hence that

$$\begin{aligned}\chi(\mathscr{L}^{\otimes -i}) &= \chi(\omega^{\otimes -mi}) \\ &= \chi(\mathcal{O}_X) + \frac{mi(mi+1)}{2} K_X^2.\end{aligned}$$

We substitute this into our expression for $\chi(f_*\mathcal{O}_Z)$ and use the well-established formulae for summing consecutive integers and their squares to obtain

$$\begin{aligned}\chi(f_*\mathcal{O}_Z) &= p\chi(\mathcal{O}_X) + \frac{K_X^2}{2}\sum_{i=0}^{p-1}(m^2 i^2 + mi) \\ &= p\chi(\mathcal{O}_X) + \frac{mp(p-1)K_X^2}{12}\left(3 + m(2p-1)\right).\end{aligned}$$

Because X and Z are each del Pezzo surfaces, Serre duality implies that $H^2(X, \mathcal{O}_X) = 0 = H^2(Z, \mathcal{O}_Z)$. Therefore, $\chi(f_*\mathcal{O}_Z) = p^e(1-q_Z)$ and $\chi(\mathcal{O}_X) = 1 - q_X$. ■

Main Theorem (2) follows as an immediate corollary:

Corollary 1.2.6. *If X is a normal, l.c.i. del Pezzo surface of degree d and irregularity $q > 0$, then there exists a nontrivial $\alpha_{\omega_X^{\otimes m}}$-torsor, Z, for which $[k_Z : k_X] = p^e$ for some integers $m \geq 1$ and $e \in \{0,1\}$. Furthermore, any such integers satisfy*

$$\begin{aligned}q &\geq 1 - \frac{1}{p^{1-e}} + \frac{md(p-1)(3 + m(2p-1))}{12} \\ &\geq \frac{d(p^2-1)}{6}, \qquad (1.2.7)\end{aligned}$$

with equality in (1.2.7) *only if $e = 1$ and $m = 1$.*

Proof. For the first inequality, use (1.2.5) of Theorem 1.2.3 and the fact $q_Z = h^1(Z, \mathcal{O}_Z) \geq 0$. For the second inequality, use $e \leq 1$ and $m \geq 1$. ■

In the case when $q_X = 1$, the values of p, m, q_Z, K_X^2, and $h^0(X, \omega_X^{-1})$ are completely determined by that of $e \in \{0, 1\}$. Later we construct examples of regular del Pezzo surfaces exhibiting these values for either choice of $e \in \{0, 1\}$ (cf. §3).

Corollary 1.2.8. *If X is a normal, l.c.i. del Pezzo surface over a field of characteristic p with irregularity $h^1(X, \mathcal{O}_X) = 1$ and Z is a nontrivial $\alpha_{\omega_X^{\otimes m}}$-torsor for an integer $m \geq 1$, then $m = 1$, $p = 2$, and the anti-canonical degree $K_X^2 = [k_Z : k_X] = 2^e$ for $e \in \{0, 1\}$. Moreover, the cohomology group $H^1(Z, \mathcal{O}_Z) = 0$, and for all $n \geq 1$,*

$$h^0(X, \omega_X^{\otimes -n}) = \frac{n(n+1)}{2^{(1-e)}}.$$

Proof. If $q_X = 1$, then the right-hand side of (1.2.5) is positive, forcing $q_Z = 0$. Thus (1.2.5) simplifies to

$$p^e = \frac{mp(p-1)K_X^2}{12} \left(3 + m(2p-1)\right).$$

As all variables are positive integers, one can quickly solve by brute force. If $e = 0$, then $p = 2, K_X^2 = 1$, and $m = 1$. Similarly, if $e = 1$, then $p = 2, K_X^2 = 2, m = 1$.

If $H^1(X, \omega_X^{\otimes n}) \neq 0$ for some $n \geq 1$, then Serre's theorem on the vanishing of higher cohomology would show the existence of some Frobenius-killed class in $H^1(X, \omega_X^{\otimes N})$, for some $N \geq n$, and then Theorem 1.2.3(2) and our above argument shows that $N = 1$. Thus, $H^1(X, \omega_X^{\otimes n}) = 0$ for all $n > 1$. By Serre duality, $h^1(X, \omega_X^{\otimes -n}) = h^1(X, \omega_X^{\otimes n+1}) = 0$ for any $n \geq 1$, and Riemann-Roch therefore implies $h^0(X, \omega_X^{\otimes -n}) = \chi(\omega_X^{\otimes -n}) = \frac{n(n+1)}{2} K_X^2$. ■

2. Algebraic foliations on regular varieties

In contrast to our task in §1 of finding numerical restrictions on the existence of regular del Pezzo surfaces with irregularity, we begin the dual problem of constructing explicit examples of such surfaces. The $\alpha_{\mathscr{L}}$-torsor construction of the previous section will again be important to us, although we shall henceforth view them from an alternative perspective. Beginning with a k-variety Z, equipped with an algebraic foliation $\mathscr{F} \subseteq T_{Z/k}$, one can construct a purely inseparable quotient morphism $f : Z \to Z/\mathscr{F}$ that factors the relative Frobenius morphism $\mathbf{F}_{Z/k} : Z \to Z \times_{k, \mathbf{F}_k} k$. If $Z \to X$ is an $\alpha_{\mathscr{L}}$-torsor, there is a natural rank 1 foliation given by the relative tangent bundle $T_{Z/X}$ that recovers X as the quotient $Z/\mathscr{F}$, for $\mathscr{F} := T_{Z/X}$. The converse does not hold as the quotient morphism $Z \to Z/\mathscr{F}$ for an arbitrary (rank 1) foliation $\mathscr{F}$ is not necessarily an $\alpha_{\mathscr{L}}$-torsor for any choice of line bundle $\mathscr{L}$ on $Z/\mathscr{F}$. However, when $p = 2$, this problem does not arise, and Z may indeed be recovered from the quotient $Z/\mathscr{F}$ as some $\alpha_{\mathscr{L}}$-torsor (cf. §2.2). Ekedahl developed this theory for smooth varieties in [2], and in this section we generalize his results to the setting of regular varieties.

Proposition 2.0.1. *Let k be a finitely generated field extension of a perfect field $\mathbb{F}_2$ of characteristic 2.*

1. *If Z is a regular k-variety and $\mathscr{F} \subseteq T_{Z/k} \subseteq T_{Z/\mathbb{F}_2}$ is a rank 1 foliation on Z over the extension $k/\mathbb{F}_2$, then the quotient $X := Z/\mathscr{F}$ is a regular k-variety and the quotient morphism $f : Z \to X$ is an $\alpha_{\mathscr{L}}$-torsor for some line bundle $\mathscr{L}$ on X.*
2. *Additionally, if Z is a del Pezzo surface and $\mathscr{F}^{\otimes 2} \cong \omega_Z$, then the quotient X is a regular del Pezzo surface, the sheaf $\mathscr{L}^{-1} \otimes \omega_X$ is a 2-torsion line bundle, and the following equations hold:*
   - *(a) $[k_Z : k_X] \cdot \chi(\mathcal{O}_Z) = 2\chi(\mathcal{O}_X) + d_X$,*
   - *(b) $K_X^2 = \frac{[k_Z : k_X] \cdot K_Z^2}{8}$.*

The fruits of our labor will be harvested in §3, when we carefully find two such foliations $\mathscr{F}_1$ and $\mathscr{F}_2$ on a specific variety Z. The resulting quotients $X_i = Z/\mathscr{F}_i$ are regular del Pezzo surfaces with irregularity $q = 1$, and in this specific case, the line bundle $\mathscr{L}$ can be identified precisely as the dualizing sheaf ω_X (cf. Cor.2.2.2).

2.1. **Quotients by foliations.** First we generalize the definition of a foliation on a smooth variety (cf. [2]) to the case of a regular variety over an imperfect field.

Definition 2.1.1. Let Z be a regular variety over a field extension k of a perfect field $\mathbb{F}$ of characteristic p. A *foliation (over the extension $k/\mathbb{F}$)* on Z is a locally free $\mathcal{O}_Z$-submodule $\mathscr{F} \subseteq T_{Z/k} \subseteq T_{Z/\mathbb{F}}$ preserved by the Lie bracket and the p-th power operation (i.e. a sub-p-Lie algebra of $T_{Z/k}$) whose cokernel $T_{Z/\mathbb{F}}/\mathscr{F}$ is locally free. The *rank* of a foliation $\mathscr{F}$ is its rank as a locally free $\mathcal{O}_Z$-module.

This definition recovers the usual notion of a foliation (cf. [2]) in the case where Z is a smooth variety over a perfect field $k = \mathbb{F}$. Our more general definition is contrived so that when $\pi : \mathcal{Z} \to \mathcal{B}$ is a morphism of varieties from a smooth variety $\mathcal{Z}$ over a perfect field $\mathbb{F}$, any foliation $\mathscr{F}$ on $\mathcal{Z}$ (in the sense of [2]) that is vertical with respect to π (i.e. $\mathscr{F} \subseteq T_{\mathcal{Z}/\mathcal{B}}$) will restrict to the generic fibre of π as a foliation (in our sense) over the extension $\mathbb{F}(\mathcal{B})/\mathbb{F}$.

The utility of algebraic foliations comes from the fact that one can use them to quotient varieties to obtain purely inseparable finite morphisms:

Definition/Lemma 2.1.2. Let $k/\mathbb{F}$ be a field extension of a perfect field $\mathbb{F}$ of characteristic p. If $\mathscr{F}$ is a foliation over the extension $k/\mathbb{F}$ on a regular k-variety Z, then there is a k-variety $Z/\mathscr{F}$, which we call the *quotient* of Z by $\mathscr{F}$, along with a purely inseparable morphism $f : Z \to Z/\mathscr{F}$ that factors the relative Frobenius morphism $\mathbf{F}_{Z/k}$ and is given locally by the inclusion of subrings $\mathcal{O}_Z^p \subseteq \mathcal{O}_{Z/\mathscr{F}} \subseteq \mathcal{O}_Z$, where

$$\mathcal{O}_{Z/\mathscr{F}} := \{f \in \mathcal{O}_Z : \delta(f) = 0 \text{ for all local derivations } \delta \in \mathscr{F}\}.$$

Proof. The construction of $Z/\mathscr{F}$ is well-defined because the definition of $\mathcal{O}_{Z/\mathscr{F}}$ commutes with localization, a result which ultimately boils down to the fact that the ring of pth powers $\mathcal{O}_Z^p$ is killed by any derivation. Since k, in addition to $\mathcal{O}_Z^p$, is killed by all derivations in $\mathscr{F} \subseteq T_{Z/k}$, the morphism f factors the relative Frobenius morphism $\mathbf{F}_{Z/k} : Z \to Z \times_{k,\mathbf{F}_k} k$. That is, $\mathbf{F}_{Z/k} = g \circ f$ for a unique morphism $g : X \to Z \times_{k,\mathbf{F}_k} k$. In particular, both f and g are purely inseparable morphisms. Moreover, since Z is finite-type over k, the relative Frobenius morphism $\mathbf{F}_{Z/k}$ is a finite morphism, and hence so are the morphisms f and g. Since Z is a finite-type over k, so is the base change $Z \times_{k,\mathbf{F}_k} k$ (with structure morphism given by projection onto the second factor). As X is finite over $Z \times_{k,\mathbf{F}_k} k$, it too is of finite type over k. ■

For foliations on smooth varieties over a perfect field $k = \mathbb{F}$, the following theorem of Ekedahl provides vital information concerning the structure of the quotient.

Theorem 2.1.3 (Ekedahl)**.** *Let Z be a smooth n-dimensional variety over a perfect field $\mathbb{F}$. Let $\mathscr{F} \subseteq T_{Z/\mathbb{F}}$ be a foliation of rank r and $f : Z \to X := Z/\mathscr{F}$ the quotient of Z by this foliation. Furthermore denote by $g : X \to Z \times_{\mathbb{F},\mathbf{F}_\mathbb{F}} \mathbb{F}$ the morphism so that $g \circ f = \mathbf{F}_{Z/\mathbb{F}}$ is the relative Frobenius morphism. Then the following hold:*

(1) X is a smooth $\mathbb{F}$-variety;
(2) f and g are finite flat morphisms of degrees p^r and p^{n-r}, respectively;
(3) there is an exact sequence

$$0 \to \mathscr{F} \to T_{Z/\mathbb{F}} \to f^* T_{X/\mathbb{F}} \to \mathbf{F}_Z^* \mathscr{F} \to 0,$$

and hence an isomorphism

$$f^*\omega_{X/\mathbb{F}} \cong \omega_{Z/\mathbb{F}} \otimes (\det \mathscr{F})^{\otimes 1-p}.$$

Proof. See [2, §3]. ■

We now partially extend this result for our applications to regular varieties over finitely generated imperfect fields.

Proposition 2.1.4. *Let Z be a regular variety over a finitely generated field extension k of a perfect field $\mathbb{F}$. Let $\mathscr{F} \subseteq T_{Z/k} \subseteq T_{Z/\mathbb{F}}$ be a foliation of rank r over the extension $k/\mathbb{F}$ and $f : Z \to X$ the quotient of Z by this foliation. Then the following hold:*

(1) X is a regular k-variety;
(2) f is a flat morphism of degree p^r;
(3) there is an exact sequence

$$0 \to \mathscr{F} \to T_{Z/\mathbb{F}} \to f^*T_{X/\mathbb{F}} \to \mathbf{F}_Z^*\mathscr{F} \to 0,$$

and hence an isomorphism

$$f^*\omega_{X/k} \cong \omega_{Z/k} \otimes (\det \mathscr{F})^{\otimes 1-p}. \tag{2.1.5}$$

Proof. Choose a sufficiently large finitely generated sub-$\mathbb{F}$-algebra $A \subseteq k$ so that Z descends to a finite-type integral A-scheme Z_A, $\mathscr{F}$ descends to a subsheaf $\mathscr{F}_A \subseteq T_{Z_A/A} \subseteq T_{Z_A/\mathbb{F}}$, and the fraction field of A equals k. This is possible because Z is of finite-type over k and $\mathscr{F}$ is a submodule of the coherent $\mathcal{O}_Z$-module $T_{Z/\mathbb{F}}$; the $\mathcal{O}_Z$-module $T_{Z/\mathbb{F}}$ is coherent because Z is of finite-type over a finitely generated field extension of $\mathbb{F}$.

It is a classical result that the regular locus of a locally Noetherian scheme is an open locus (cf. [12, Thm. 24.4]). Since $Z = Z_A \times_A k$ is regular, the regular locus on Z_A is a non-empty open neighborhood of the generic fibre Z, and its image in A will be an open neighborhood U of the generic point of A. By replacing $\operatorname{Spec} A$ by a sufficiently small affine subset of U, we may assume that both Z_A and $\operatorname{Spec} A$ are regular $\mathbb{F}$-varieties. Consequently, both Z_A and $\operatorname{Spec} A$ are smooth over $\mathbb{F}$ since $\mathbb{F}$ is a perfect field (cf. [18, Lem. 038V]).

Because $\mathscr{F}$ is a foliation, $\mathscr{F} \cong \mathscr{F}_A \otimes_A k$ and $T_{Z/\mathbb{F}}/\mathscr{F} \cong (T_{Z_A/\mathbb{F}}/\mathscr{F}_A) \otimes_A k$ are locally free. Therefore, by replacing $\operatorname{Spec} A$ by an even smaller open subscheme, we may assume that both $\mathscr{F}_A$ and $T_{Z_A/\mathbb{F}}/\mathscr{F}_A$ are finite locally free $\mathcal{O}_A$-modules. Consider the $\mathcal{O}_A$-module homomorphism $\mathscr{F}_A \otimes \mathscr{F}_A \to T_{Z_A/\mathbb{F}}/\mathscr{F}_A$ induced by the Lie bracket and the $\mathcal{O}_A$-module homomorphism $\mathscr{F}_A \to T_{Z_A/\mathbb{F}}/\mathscr{F}_A$ induced by the pth power operation. Notice that both of these homomorphisms are 0 when localized at the generic point of $\operatorname{Spec} A$ precisely because of our hypothesis that $\mathscr{F}$ is a foliation on the generic fibre Z. By the upper semi-continuity of rank, we may restrict $\operatorname{Spec} A$ even further so that these morphisms are 0 over all of $\operatorname{Spec} A$, which is equivalent to $\mathscr{F}_A$ being a foliation on the smooth variety Z_A over $\mathbb{F}$.

Now, we may apply Theorem 2.1.3 to Z_A and $\mathscr{F}_A \subseteq T_{Z_A/A} \subseteq T_{Z_A/\mathbb{F}}$ to obtain a smooth quotient $X_A := Z_A/\mathscr{F}_A$. The generic fibre $X_A \times_A k$ is therefore a regular variety. Because taking quotients by foliations is a local operation, $Z/\mathscr{F} \cong X_A \times_A k$, proving (1). Assertion (2) holds by localizing the morphism $f : Z_A \to X_A$, which is finite and flat of degree r by Theorem 2.1.3. The exact sequence in (3) follows by localizing that of Theorem 2.1.3(3). The isomorphism in (3) follows by taking the determinant of this sequence, which yields

$$f^*\omega_{X_A/\mathbb{F}}|_Z \cong \omega_{Z_A/\mathbb{F}}|_Z \otimes (\det \mathscr{F})^{\otimes 1-p},$$

and then applying Lemma 2.1.6 to each of the morphisms $Z_A \to A$ and $X_A \to A$. ■

Lemma 2.1.6. *Let $\pi : \mathcal{X} \to \mathcal{B}$ be an l.c.i. morphism of l.c.i. varieties over a field $\mathbb{F}$. Let $k := \mathbb{F}(\mathcal{B})$ denote the function field of $\mathcal{B}$. Then the dualizing sheaf of the generic fibre $X := \mathcal{X} \times_{\mathcal{B}} k$ is just the restriction of the dualizing sheaf of $\mathcal{X}$:*

$$\omega_{X/k} = \omega_{\mathcal{X}/\mathbb{F}}|_X.$$

Proof. By [6, Def. 1.5], we have $\omega_\pi := \omega_{\mathcal{X}/\mathbb{F}} \otimes \pi^*\omega_{\mathcal{B}/\mathbb{F}}^{-1}$. As ω_π commutes with arbitrary base changes,

$$\omega_{X/k} = \omega_\pi|_X = (\omega_{\mathcal{X}/\mathbb{F}} \otimes \pi^*\omega_{\mathcal{B}/\mathbb{F}}^{-1})|_X = \omega_{\mathcal{X}/\mathbb{F}}|_X,$$

with the last equality justified by $\omega_{\mathcal{B}/\mathbb{F}}$ being locally trivial on $\mathcal{B}$. ∎

2.2. **Foliations in characteristic two.** A degree p inseparable morphism $f : Z \to X$ is not generally an $\alpha_{\mathscr{L}}$-torsor for any line bundle $\mathscr{L}$, even when f is the morphism arising from the quotient by a foliation on a variety Z. Luckily, when $p = 2$ this difficulty does not arise, which allows us to apply Theorem 1.2.3 to the proof of Proposition 2.0.1, the key result used in §3 to construct regular del Pezzo surfaces with irregularity.

Proposition 2.2.1 (Ekedahl)**.** *Let $f : Z \to X$ be a finite morphism of degree $p = 2$ from a Cohen-Macaulay scheme Z to a regular variety X. Let $\mathscr{L}$ be the line bundle satisfying*

$$0 \to \mathcal{O}_X \to f_*\mathcal{O}_Z \to \mathscr{L}^{-1} \to 0.$$

Then $Z \to X$ is an α_s torsor for some $s \in \Gamma(X, \mathscr{L})$, viewed as a section $s \in \mathrm{Hom}(\mathscr{L}, \mathscr{L}^{\otimes 2})$, where α_s is the group scheme kernel of $\mathbf{F}_{\mathscr{L}/X} - s : \mathscr{L} \to \mathscr{L}^{\otimes 2}$. Moreover, if f is a purely inseparable map, then $s = 0$ and hence $f : Z \to X$ is an $\alpha_{\mathscr{L}}$-torsor.

Proof. This is proven in [3, Prop. 1.11] for smooth X, although the proof only requires X to be regular (to guarantee that $f_*\mathcal{O}_Z$ is a locally free $\mathcal{O}_X$-module). ∎

We now prove the result advertised at the beginning of this section:

Proof of Proposition 2.0.1. Proposition 2.1.4 (1) proves that X is regular, and then Theorem 2.2.1 shows that $f : Z \to X$ indeed arises as an $\alpha_{\mathscr{L}}$-torsor. Proposition 2.1.4(3) proves $f^*\omega_X \cong \omega_Z \otimes \mathscr{F}^{\otimes -1}$ and Proposition 1.1.2 gives $f^*\mathscr{L} \cong \omega_Z \otimes f^*\omega_X^{-1}$. It immediately follows $\mathscr{F} \cong f^*\mathscr{L}$, and also $f^*\omega_X \cong \mathscr{F}$, due to the hypothesis $\omega_Z \cong \mathscr{F}^{\otimes 2}$. Combining these isomorphisms, we obtain $f^*\omega_X \cong f^*\mathscr{L}$. Since f is a finite, flat surjective map of degree 2, the line bundles ω_X and $\mathscr{L}$ differ by a 2-torsion line bundle, and hence are $\mathbb{Q}$-linearly equivalent. If Z is a del Pezzo surface, then X is as well because f is a finite, flat surjective map and so ω_X^{-1} is ample if and only if $\omega_Z^{-1} \cong f^*\omega_X^{\otimes -2}$ is ample. A straight-forward application of Theorem 1.2.3 (2) gives the last two claims. ∎

Corollary 2.2.2. *If $Z \to X$ is the quotient of a regular del Pezzo surface Z by a rank 1 foliation $\mathscr{F}$ over $k/\mathbb{F}$, a finitely generated field extension of a perfect field, such that $\mathscr{F}^{\otimes 2} \cong \omega_Z$ and $h^1(X, \mathcal{O}_X) = 1$, then Z is an $\alpha_{\mathscr{L}}$-torsor for $\mathscr{L} \cong \omega_X$.*

Proof. Corollary 1.2.8 guarantees that $\mathbb{F}$ is of characteristic 2. By Proposition 2.0.1, Z is a nontrivial $\alpha_{\mathscr{L}}$-torsor for some line bundle $\mathscr{L}$ that differs from ω_X by a 2-torsion line bundle. In particular, $\mathscr{L}$ and ω_X are numerically equivalent, and therefore by the Riemann-Roch theorem, $\chi(\mathscr{L}) = \chi(\omega_X)$. Serre duality implies $\chi(\omega_X) = \chi(\mathcal{O}_X) = 0$, because $h^1(X, \mathcal{O}_X) = 1$. The groups $H^0(X, \mathscr{L})$ and $H^0(X, \mathscr{L}^{\otimes 2})$ are 0 because $\mathscr{L}^{-1}$ is ample. Therefore, $h^1(X, \mathscr{L}) = h^2(X, \mathscr{L})$, and by Serre duality, $h^2(X, \mathscr{L}) = h^0(X, \mathscr{L}^{-1} \otimes \omega_X)$.

If we assume $\mathscr{L}^{-1} \otimes \omega_X$ is a nontrivial line bundle, then it follows that

$$h^1(X, \mathscr{L}) = h^0(X, \mathscr{L}^{-1} \otimes \omega_X) = 0,$$

because any global section of a nontrivial torsion line bundle on a projective variety is 0. On the other hand, since Z is a nontrivial $\alpha_{\mathscr{L}}$-torsor, it corresponds to a nonzero class of the cohomology group $H^1(X, \alpha_{\mathscr{L}})$. The long exact sequence in cohomology attached to the short exact sequence of group schemes

$$0 \to \alpha_{\mathscr{L}} \to \mathscr{L} \to \mathscr{L}^{\otimes 2} \to 0,$$

along with the vanishing $H^0(X, \mathscr{L}^{\otimes 2}) = 0$, proves that there is an injection $H^1(X, \alpha_{\mathscr{L}}) \subseteq H^1(X, \mathscr{L})$. This latter group is 0, yet must have a nonzero class that corresponds to the nontrivial $\alpha_{\mathscr{L}}$-torsor Z, demonstrating the absurdity of our assumption. Therefore $\mathscr{L}^{-1} \otimes \omega_X \cong \mathcal{O}_X$. ■

3. The construction of regular del Pezzo surfaces with irregularity

In this section we construct examples of regular del Pezzo surfaces X with $h^1(X, \mathcal{O}_X) = 1$. By Corollary 1.2.8, these surfaces can only exist in characteristic 2 and must have anti-canonical degree $K_X^2 \in \{1, 2\}$. We construct such surfaces by applying Proposition 2.0.1 to an explicit regular del Pezzo surface Z and foliations $\mathscr{F}$ satisfying $\mathscr{F}^{\otimes 2} \cong \omega_Z$. Once constructed, it follows from Corollary 2.2.2 that $Z \to X$ is an α_{ω_X}-torsor.

3.1. **Set-up.** Let $\mathbb{F}_2$ be any perfect field of characteristic 2. Let $\mathcal{Z} \subseteq \mathbb{P}^3_{\mathbb{F}_2} \times \mathbb{A}^4_{\mathbb{F}_2}$ be the family of quasi-linear quadrics given by the vanishing of the form $Q := \alpha_0 X_0^2 + \alpha_1 X_1^2 + \alpha_2 X_2^2 + \alpha_3 X_3^2$, where the coordinates $[X_0 : X_1 : X_2 : X_3]$ are projective and $(\alpha_0, \alpha_1, \alpha_2, \alpha_3)$ are affine. As a simplification, we sometimes omit the symbol $\mathbb{F}_2$ from our notation (e.g. we write $\Omega_{\mathbb{P}^3}$ instead of the more cluttered $\Omega_{\mathbb{P}^3_{\mathbb{F}_2}/\mathbb{F}_2}$). Let $\mathscr{I}_{\mathcal{Z}} \subseteq \mathcal{O}_{\mathbb{P}^3 \times \mathbb{A}^4}$ denote the ideal sheaf, generated by Q, that defines $\mathcal{Z}$ as a subscheme of $\mathbb{P}^3_{\mathbb{F}_2} \times \mathbb{A}^4_{\mathbb{F}_2}$.

The sequence

$$0 \to \mathcal{O}_{\mathcal{Z}}(-2) \stackrel{dQ}{\to} \mathcal{O}_{\mathcal{Z}} \otimes (\Omega^1_{\mathbb{P}^3} \oplus \Omega^1_{\mathbb{A}^4}) \to \Omega^1_{\mathcal{Z}/\mathbb{F}_2} \to 0 \tag{3.1.1}$$

is exact, and since $dQ = \sum X_i^2 d\alpha_i$ is nowhere vanishing, the cokernel $\Omega_{\mathcal{Z}/\mathbb{F}_2}$ is a rank 6 vector bundle on $\mathcal{Z}$. Hence, $\mathcal{Z}$ is a smooth $\mathbb{F}_2$-variety. Let $\mathcal{Z}_U$ denote the restriction of the family $\mathcal{Z}$ to the open subscheme $U \subseteq \mathbb{A}^4_{\mathbb{F}_2}$ that complements the 15 hyperplanes of the form $\sum_{i=0}^3 \varepsilon_i \alpha_i = 0$ for $\varepsilon_i \in \{0, 1\}$. Let Z be the generic fibre of $\mathcal{Z}_U$ over U,

$$Z := (\sum \alpha_i X_i^2 = 0) \subseteq \mathbb{P}^3_{\mathbb{F}_2(\alpha_0, \ldots, \alpha_3)}.$$

The adjunction formula implies $\omega_Z \cong \mathcal{O}_Z(-2)$, and hence Z is a regular del Pezzo surface with $K_Z^2 = 8$ and, being a hypersurface in $\mathbb{P}^3_{\mathbb{F}_2(\alpha_0, \ldots, \alpha_3)}$, with $h^1(Z, \mathcal{O}_Z) = 0$.

To satisfy the hypothesis of Proposition 2.0.1, we shall construct, for specified subfields $k \subseteq \mathbb{F}_2(\alpha_0, \ldots, \alpha_3)$, rank 1 foliations $\mathcal{O}_Z(-1) \cong \mathscr{F} \subseteq T_{Z/k} \subseteq T_{Z/\mathbb{F}_2}$ over the extension $k/\mathbb{F}_2$. To construct such $\mathscr{F}$, we find subsheaves $\mathscr{F}_{\mathcal{Z}} \subseteq T_{\mathcal{Z}/\mathbb{F}_2}$ that restrict to foliations on $\mathcal{Z}_U$ over the perfect field $\mathbb{F}_2$. We then take $\mathscr{F}$ to be the restriction of this foliation to Z, that is, $\mathscr{F} := (\mathscr{F}_{\mathcal{Z}})|_Z$.

3.2. **Example of degree one.** Define $\Theta_{\mathbb{P}} : \mathcal{O}_{\mathbb{P}^3}(-1) \to T_{\mathbb{P}^3}$ as the composition

$$\Theta_{\mathbb{P}} : \mathcal{O}_{\mathbb{P}^3}(-1) \stackrel{\sum X_i^2 \partial_{X_i}}{\longrightarrow} \sum_{i=0}^{3} \mathcal{O}_{\mathbb{P}^3}(1) \partial_{X_i} \stackrel{\phi}{\longrightarrow} T_{\mathbb{P}^3},$$

where ϕ is the morphism coming from the Euler sequence,

$$0 \to \mathcal{O}_{\mathbb{P}^3_{\mathbb{F}_2}} \stackrel{\sum X_i \partial_{X_i}}{\longrightarrow} \sum_{i=0}^{3} \mathcal{O}_{\mathbb{P}^3_{\mathbb{F}_2}}(1) \partial_{X_i} \stackrel{\phi}{\longrightarrow} T_{\mathbb{P}^3} \to 0. \tag{3.2.1}$$

Let $\mathscr{F}_{\mathcal{Z}}$ denote the image of

$$\Theta_{\mathbb{P}} \oplus 0 : \mathcal{O}_{\mathcal{Z}} \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \to \mathcal{O}_{\mathcal{Z}} \otimes (T_{\mathbb{P}^3} \oplus T_{\mathbb{A}^4}) \cong T_{\mathbb{P}^3 \times \mathbb{A}^4}|_{\mathcal{Z}}.$$

Notice that, as derivations in $\mathscr{F}_{\mathcal{Z}}$ preserve the ideal sheaf $\mathscr{I}_{\mathcal{Z}}$, as well as kill functions coming from $\mathcal{O}_{\mathbb{A}^4}$, the sheaf $\mathscr{F}_{\mathcal{Z}}$ is contained within the subsheaf $T_{\mathcal{Z}/\mathbb{A}^4} \subseteq T_{\mathbb{P}^3 \times \mathbb{A}^4}|_{\mathcal{Z}}$.

We now proceed to demonstrate that $\mathscr{F}_{\mathcal{Z}} \subseteq T_{\mathcal{Z}/\mathbb{F}_2}$ is foliation over $\mathbb{F}_2$ when restricted to $\mathcal{Z}_U$. First we prove that $\Theta_{\mathbb{P}} \oplus 0$ is injective on all fibres over $\mathcal{Z}_U$. It suffices to prove this injectivity after composing with the projection $T_{\mathbb{P}^3 \times \mathbb{A}^4}|_{\mathcal{Z}} \to \mathcal{O}_{\mathcal{Z}} \otimes T_{\mathbb{P}^3}$. In view of (3.2.1), this composition fails to be injective precisely over the points where $\sum X_i^2 \partial_{X_i}$ and $\sum X_i \partial_{X_i}$ fail to span a 2-dimensional subspace of the fibre of $\sum_{i=0}^{3} \mathcal{O}_{\mathbb{P}^3_{\mathbb{F}_2}}(1)\partial_{X_i}$, which exactly constitutes the vanishing of all 2×2 minors of the matrix:

$$\begin{bmatrix} X_0^2 & X_1^2 & X_2^2 & X_3^2 \\ X_0 & X_1 & X_2 & X_3 \end{bmatrix}.$$

Such minors are of the form $X_i X_j (X_i + X_j)$, for $i \neq j$, and one quickly checks that they cannot simultaneously vanish on $\mathcal{Z}_U$.

As $\mathscr{F}_{\mathcal{Z}}$ is rank 1, it is preserved under the Lie bracket, and the only remaining criterion $\mathscr{F}_{\mathcal{Z}}$ must satisfy is closure under pth powers. It suffices to verify this condition on a local generator of $\mathscr{F}_{\mathcal{Z}}$. On the chart $(X_{i_0} \neq 0)$, the sheaf $\mathscr{F}_{\mathcal{Z}}$ is generated by the differential operator

$$\theta_{\mathbb{P}} := \Theta_{\mathbb{P}}(\frac{1}{X_{i_0}}) = \frac{1}{X_{i_0}} \sum X_i^2 \partial_{X_i}.$$

If $x_i := \frac{X_i}{X_{i_0}}$ are the local affine coordinates, then

$$\theta_{\mathbb{P}} = \sum_{i \neq i_0} (x_i^2 + x_i)\partial_{x_i},$$

because $X_i \partial_{X_i} = x_i \partial_{x_i}$, for $i \neq i_0$, and $X_{i_0} \partial_{X_{i_0}} = \sum_{i \neq i_0} x_i \partial_{x_i}$, as can be checked by evaluation on the functions $x_j = \frac{X_j}{X_{i_0}}$. We now expand $\theta_{\mathbb{P}}^2$, taking note that all higher-order operators in the expansion are either 0 (e.g. $\partial_{x_i}^2 = 0$) or are nonzero (e.g. $\partial_{x_i}\partial_{x_j}$, $i \neq j$) but occur with even, hence 0, coefficient:

$$\begin{aligned} \theta_{\mathbb{P}}^2 &= \sum_{i \neq i_0} (x_i^2 + x_i)\partial_{x_i} \circ \sum_{j \neq i_0} (x_j^2 + x_j)\partial_{x_j} \\ &= \sum_{i \neq i_0} (x_i^2 + x_i) \sum_{j \neq i_0} \delta_{ij} \cdot \partial_{x_j} \\ &= \theta_{\mathbb{P}}. \end{aligned}$$

Thus, $\mathscr{F}_{\mathcal{Z}} \subseteq T_{\mathcal{Z}/\mathbb{A}^4}$ is a foliation on $\mathcal{Z}_U$, and the restriction $\mathscr{F} := (\mathscr{F}_{\mathcal{Z}})|_Z$ is therefore a foliation over the extension $\mathbb{F}_2(\alpha_0, \dots, \alpha_3)/\mathbb{F}_2$. Let $X_1 := Z/\mathscr{F}$ be the resulting quotient.

Theorem 3.2.2. *The variety X_1 constructed above is a regular del Pezzo surface over the field $H^0(X_1, \mathcal{O}_{X_1}) = \mathbb{F}_2(\alpha_0, \alpha_1, \alpha_2, \alpha_3)$ with irregularity $h^1(X_1, \mathcal{O}_{X_1}) = 1$ and degree $K_{X_1}^2 = 1$.*

Proof. The variety Z defined above is a regular del Pezzo surface with $d_Z = 8$, $\chi(\mathcal{O}_Z) = 1$. Because $H^0(Z, \mathcal{O}_Z) = \mathbb{F}_2(\alpha_0, \dots, \alpha_3)$ and X_1 is an $\mathbb{F}_2(\alpha_0, \dots, \alpha_3)$-variety, $H^0(X_1, \mathcal{O}_{X_1}) = \mathbb{F}_2(\alpha_0, \dots, \alpha_3)$ as well. Proposition 2.0.1 therefore applies with $[k_Z : k_{X_1}] = 1$, proving the theorem. ∎

Remark 3.2.3. *Actually, there exists a regular del Pezzo surface X_1' of degree and irregularity 1 defined over the subfield $\mathbb{F}_2(\alpha_0, \alpha_1, \alpha_2) \subseteq \mathbb{F}_2(\alpha_0, \alpha_1, \alpha_2, \alpha_3)$. Indeed, the closed subscheme $\mathcal{Z}_{U \cap \mathbb{A}^3} \subseteq \mathcal{Z}_U$ sitting over the inclusion $U \cap \mathbb{A}^3 \subseteq U$ given by $\alpha_3 = 1$ is smooth. The foliation $\mathscr{F}_{\mathcal{Z}}|_{\mathcal{Z}_U}$ restricts to a foliation on $\mathcal{Z}_{U\cap\mathbb{A}^3}$, and the quotient of the generic fibre by this foliation is the desired surface X_1'. Any subvariety $B \subseteq U$ of dimension strictly less than 3 gives rise to a singular closed subscheme $\mathcal{Z}_{U\cap B} \subseteq \mathcal{Z}_U$, and here our method breaks down.*

3.3. **Example of degree two.** Let $\mathcal{Z}_U \to U$ be the family defined in §3.1. We again choose $\mathscr{F}_{\mathcal{Z}} \cong \mathcal{O}_{\mathcal{Z}}(-1)$, but this time to be the subsheaf of $T_{\mathcal{Z}/k}$ defined by the image of

$$\Theta_{\mathbb{P}} \oplus \Theta_{\mathbb{A}} : \mathcal{O}_{\mathcal{Z}}(-1) \to \mathcal{O}_{\mathcal{Z}} \otimes (T_{\mathbb{P}^3} \oplus T_{\mathbb{A}^4}) \cong T_{\mathbb{P}^3\times\mathbb{A}^4}|_{\mathcal{Z}},$$

for $\Theta_{\mathbb{P}} = \sum X_i^2 \partial_{X_i}$ as before, and $\Theta_{\mathbb{A}} := (\sum X_j) \sum_k \alpha_k \partial_{\alpha_k}$. Again, we work locally on the chart $(X_{i_0} \neq 0)$, with affine coordinates $x_i := \frac{X_i}{X_{i_0}}$. A local generator of $\mathscr{F}_{\mathcal{Z}}$ is given by $\theta = \theta_{\mathbb{P}} + \theta_{\mathbb{A}}$, for $\theta_{\mathbb{P}} = \sum_{i\neq i_0}(x_i + x_i^2)\partial_{x_i}$ and $\theta_{\mathbb{A}} = (1 + \sum_{i\neq i_0} x_i) \sum \alpha_j \partial_{\alpha_j}$. We saw above that the image of $\Theta_{\mathbb{P}}$ preserves the ideal sheaf $\mathscr{I}_{\mathcal{Z}}$, and therefore is contained within $T_{\mathcal{Z}/\mathbb{F}_2}$. We now check that the image of $\Theta_{\mathbb{A}}$ does as well. On the chart $(X_{i_0} \neq 0)$, the ideal $\mathscr{I}_{\mathcal{Z}}$ is generated by

$$q := \frac{1}{X_{i_0}^2} \cdot Q = \alpha_{i_0} + \sum_{i\neq i_0} \alpha_i x_i^2.$$

As $\theta_{\mathbb{A}}(q) = (1 + \sum_{i\neq i_0} x_i)q = 0$, the image of $\Theta_{\mathbb{A}}$ preserves the ideal sheaf, and therefore the image of $\Theta_{\mathbb{P}} + \Theta_{\mathbb{A}}$ is contained in $T_{\mathcal{Z}/\mathbb{F}_2}$, that is, $\mathscr{F}_{\mathcal{Z}} \subseteq T_{\mathcal{Z}/\mathbb{F}_2}$.

We next begin to show that the subsheaf $\mathscr{F}_{\mathcal{Z}} \subseteq T_{\mathcal{Z}/\mathbb{F}_2}$ is a foliation over $\mathbb{F}_2$ on $\mathcal{Z}$. In the previous subsection, we showed that $\Theta_{\mathbb{P}}$ is injective on fibres over $\mathcal{Z}_U$, and it immediately follows that the same is true of the sum $\Theta_{\mathbb{P}} \oplus \Theta_{\mathbb{A}}$. Hence $\mathscr{F}_{\mathcal{Z}}$ is a subbundle of $T_{\mathcal{Z}/\mathbb{F}_2}$. The Lie bracket preserves $\mathscr{F}_{\mathcal{Z}}$ simply because $\mathscr{F}_{\mathcal{Z}}$ is rank 1, and as before, our final verification is whether $\mathscr{F}_{\mathcal{Z}}$ is closed under squaring. The following local calculation shows just that:

$$\begin{aligned}
\theta^2 &= (\theta_{\mathbb{P}} + \theta_{\mathbb{A}})^2 \\
&= \theta_{\mathbb{P}}^2 + \theta_{\mathbb{P}} \circ \theta_{\mathbb{A}} + \theta_{\mathbb{A}} \circ \theta_{\mathbb{P}} + \theta_{\mathbb{A}}^2 \\
&= \theta_{\mathbb{P}} + (\sum_{i\neq i_0} x_i + x_i^2)(\sum \alpha_j \partial_{\alpha_j}) + 0 + (1 + \sum_{i\neq i_0} x_i)^2 (\sum \alpha_j \partial_{\alpha_j}) \\
&= \theta_{\mathbb{P}} + \theta_{\mathbb{A}} = \theta.
\end{aligned}$$

This proves that $\mathscr{F}_{\mathcal{Z}}$ is a foliation on $\mathcal{Z}_U$ over $\mathbb{F}_2$. Let $\mathscr{F} := (\mathscr{F}_{\mathcal{Z}})|_Z$ be the restriction of this foliation to Z. If $k := \mathbb{F}_2(\alpha_i\alpha_j : 0 \leq i,j \leq 3) \subseteq \mathbb{F}_2(\alpha_0, \ldots, \alpha_3)$, then $\mathscr{F} \subseteq T_{Z/k}$, since the image of both $\Theta_{\mathbb{P}}$ and $\Theta_{\mathbb{A}}$ kills all elements of k. Let $X_2 := Z/\mathscr{F}$ be the resulting quotient k-variety.

Theorem 3.3.1. *The variety X_2 constructed above is a regular del Pezzo surface over the field $H^0(X_2, \mathcal{O}_{X_2}) = \mathbb{F}_2(\alpha_i\alpha_j : 0 \leq i,j \leq 3) \subseteq \mathbb{F}_2(\alpha_0, \ldots, \alpha_3)$ with irregularity $h^1(X_2, \mathcal{O}_{X_2}) = 1$ and degree $K_{X_2}^2 = 2$.*

Proof. We reiterate that Z is a regular del Pezzo surface with $K_Z^2 = 8, \chi(\mathcal{O}_Z) = 1$, and $H^0(Z, \mathcal{O}_Z) = \mathbb{F}_2(\alpha_0, \ldots, \alpha_3)$. The variety X_2 is defined over the field $k = \mathbb{F}_2(\alpha_i\alpha_j : 0 \leq i,j \leq 3)$, and therefore $k \subseteq H^0(X_2, \mathcal{O}_{X_2}) \subseteq H^0(Z, \mathcal{O}_Z)$. The foliation $\mathscr{F}$ does not kill all of $H^0(Z, \mathcal{O}_Z)$, since $\theta(\alpha_0) = \alpha_0(1 + \sum_{i\neq i_0} x_i) \neq 0$. Hence, α_0 is not contained in $H^0(X_2, \mathcal{O}_{X_2})$, which is therefore a proper subfield of $H^0(Z, \mathcal{O}_Z)$ containing k. As k is of index 2 in $H^0(Z, \mathcal{O}_Z)$, the fields $H^0(X_2, \mathcal{O}_{X_2})$ and k must coincide. We conclude by applying Proposition 2.0.1 with $[k_Z : k_{X_2}] = 2$. ■

3.4. **Geometric reducedness.** We conclude this section by proving that, of our examples constructed above, the surface of degree 1 is geometrically reduced while the surface of degree 2 is geometrically non-reduced.

Proposition 3.4.1. *The regular del Pezzo surface X_1 is geometrically reduced, but the regular del Pezzo surface X_2 is geometrically non-reduced.*

Proof. Let $k_i := H^0(X_i, \mathcal{O}_{X_i})$ denote the field of global function on X_i, for $i \in \{1, 2\}$. We will begin with the case of $i = 1$, and we will use the notation established in §3.2. Since X_1 is Cohen-Macaulay, it is geometrically reduced if and only if it is so generically, and thus suffices to prove X_1 is geometrically reduced on the affine chart over which

$$\mathcal{O}_{Z_{\bar{k}_1}} = \bar{k}_1[x_1, x_2, x_3]/\ell^2, \quad \text{with } \ell := \sqrt{\alpha_0} + \sqrt{\alpha_1} \cdot x_1 + \sqrt{\alpha_2} \cdot x_2 + \sqrt{\alpha_3} \cdot x_3.$$

The ring $R := \mathcal{O}_{(X_1)_{\bar{k}_1}}$ is the subring of $\mathcal{O}_{Z_{\bar{k}_1}}$ on which the differential $\theta_{\mathbb{P}} := \sum_{i \neq 0}(x_i + x_i^2)\partial_{x_i}$ vanishes.

For the purpose of proving that R is reduced, assume $f \in \bar{k}_1[x_1, x_2, x_3]$ lifts a nilpotent element of R. This means that, in the ring $\bar{k}_1[x_1, x_2, x_3]$, the polynomial $\theta_{\mathbb{P}}(f)$ is divisible by ℓ, and secondly, for some $n > 0$, the quadratic form ℓ^2 divides f^n, which by unique factorization implies that $f = \ell \cdot g$ for some polynomial g. Consequently, ℓ divides the product $\theta_{\mathbb{P}}(\ell) \cdot g$ due to the Leibnitz rule:

$$\theta_{\mathbb{P}}(f) = \theta_{\mathbb{P}}(\ell) \cdot g + \ell \cdot \theta_{\mathbb{P}}(g).$$

We can compute $\theta_{\mathbb{P}}(\ell)$ explicitly as

$$\begin{aligned} \theta_{\mathbb{P}}(\ell) &= \sum_{i=1}^{3} (x_i + x_i^2) \frac{\partial f}{\partial x_i} \\ &= \ell + (\sqrt{\alpha_0} + \sum_{i=1}^{3} \sqrt{\alpha_i} \cdot x_i^2). \end{aligned}$$

Consider the morphism $\bar{k}_1[x_1, x_2, x_3]/\ell \twoheadrightarrow \bar{k}_1$ defined by $x_1, x_2 \mapsto 0$, and $x_3 \mapsto \sqrt{\alpha_0/\alpha_3}$. This morphism sends $\theta_{\mathbb{P}}(\ell) \mapsto \sqrt{\alpha_0} + \alpha_0/\sqrt{\alpha_3} \neq 0$, and so ℓ does not divide $\theta_{\mathbb{P}}(\ell)$. Therefore, ℓ must divide g, and hence ℓ^2 divides f, which implies the image of f in R was 0 to begin with. Thus R is reduced.

Now, consider $f : Z \to X_2$, as in §3.3. Let $k_2 := H^0(X_2, \mathcal{O}_{X_2})$ and $k_2' := H^0(Z, \mathcal{O}_Z)$. As the degree of the field extension is $[k_2' : k_2] = 2$, and Z is geometrically a first-order neighborhood of a plane, the variety $Z \times_{k_2} \bar{k}_2$ has generic point $\bar{\xi}_Z$ whose local ring $k_2'(Z) \otimes_{k_2} \bar{k}_2$ is Artinian of length 4. If X_2 were geometrically reduced, then $k_2(X_2) \otimes_{k_2} \bar{k}_2$ would be a field, and $k_2'(Z) \otimes_{k_2} \bar{k}_2$ a 2-dimensional vector space over this field, with length at most 2, yielding a contradiction. ■

4. A GEOMETRIC DESCRIPTION OF THE SURFACE OF DEGREE ONE

In this section we study, through explicit computation, the regular del Pezzo surface X_1 over the field $\mathbb{F}_2(\alpha_0, \alpha_1, \alpha_2, \alpha_3)$ constructed in §3.2. Although Remark 3.2.3 asserts that there exists an analogous regular del Pezzo surface X_1' defined over the subfield $\mathbb{F}_2(\alpha_0, \alpha_1, \alpha_2)$, for the sake of symmetry in our calculations, we will restrict our attention to the surface X_1, which for convenience we will henceforth denote by X.

The surface X is geometrically integral and is of anti-canonical degree and irregularity one: $K_X^2 = 1$, $h^1(X, \mathcal{O}_X) = 1$. By Reid's classification of non-normal del Pezzo surfaces [15], the normalization of the geometric base change $X_{\bar{k}}$ is isomorphic to the projective plane, $X_{\bar{k}}^{\nu} \cong \mathbb{P}_{\bar{k}}^2$, and

the normalization morphism consists of the collapse of a double line onto a cuspidal curve $C \subseteq X_{\bar{k}}$ of arithmetic genus $h^1(C, \mathcal{O}_C) = 1$. The upshot of our calculations is a concrete realization of this description of $X_{\bar{k}}$ in terms of our construction of X as the quotient by a foliation:

Proposition 4.0.1. *Let $k := \mathbb{F}_2(\alpha_0, \alpha_1, \alpha_2, \alpha_3)$ and $Z \to X$ denote the quotient morphism from the regular variety $Z := (\sum \alpha_i X_i^2 = 0) \subseteq \mathbb{P}^3_k$, defined by the foliation described in §3.2.*

(1) The reduced scheme $Z^{red}_{\bar{k}}$ is the hyperplane $(\sum \sqrt{\alpha_i} X_i = 0) \subseteq \mathbb{P}^3_{\bar{k}}$, and the induced morphism $Z^{red}_{\bar{k}} \to X_{\bar{k}}$ is the normalization of the variety $X_{\bar{k}}$.
(2) The singular locus of $X_{\bar{k}}$ is a rational cuspidal curve C of arithmetic genus one.
(3) The inverse image of C in $Z^{red}_{\bar{k}}$ is the double line D described by the equation

$$(\sum \sqrt[4]{\alpha_i} X_i)^2 = 0.$$

(4) The cusp of C sits below the unique point on D satisfying the additional equation

$$\sum \sqrt[8]{\alpha_i} X_i = 0.$$

This is proven in stages throughout the following subsections.

4.1. **Normalization of geometric base change.** We recall the notation established in §3.2. The variety $Z := (\sum_{i=0}^3 \alpha_i X_i^2 = 0) \subseteq \mathbb{P}^3_k$ is a regular del Pezzo surface over the field $k := \mathbb{F}_2(\alpha_0, \alpha_1, \alpha_2, \alpha_3)$, and $\mathscr{F} = \operatorname{Im}(\Theta_{\mathbb{P}}) \subseteq T_{Z/k}$ is the foliation on Z over the extension $k/\mathbb{F}_2$ defined by $\Theta_{\mathbb{P}} := \sum_{i=0}^3 X_i^2 \partial_{X_i}$. Recall that X was defined as the quotient $X = Z/\mathscr{F}$ and as before $f : Z \to X$ will denote the quotient morphism.

Proposition 4.1.1. *The relative Frobenius morphism $\mathbf{F}_{Z/k}$ factors as*

(4.1.2)
$$\begin{tikzcd} & Z \arrow[d, "f"] \arrow[dl, "\bar{\mathbf{F}}_{Z/k}"'] \arrow[dd, bend left, "\mathbf{F}_{Z/k}"] \\ & X \arrow[d, "g"] \arrow[dl, "\bar{g}"'] \\ (Z \times_{k,\mathbf{F}_k} k)^{red} \arrow[r] & Z \times_{k,\mathbf{F}_k} k, \end{tikzcd}$$

with morphisms $\bar{\mathbf{F}}_{Z/k}$, $\bar{g}$, and f flat and finite with respective degrees $8, 4$, and 2. The geometric base change of the top triangle admits a further factorization,

(4.1.3)
$$\begin{tikzcd} Z^{red}_{\bar{k}} \arrow[r] \arrow[d, "h"'] \arrow[dr, "\bar{f}_{\bar{k}}"] & Z_{\bar{k}} \arrow[d, "f_{\bar{k}}"'] \\ (Z \times_{k,\mathbf{F}_k} \bar{k})^{red} & X_{\bar{k}}, \arrow[l, "\bar{g}_{\bar{k}}"'] \end{tikzcd}$$

where the morphism $\bar{f}_{\bar{k}} : Z^{red}_{\bar{k}} \to X_{\bar{k}}$ identifies $Z^{red}_{\bar{k}} \cong \mathbb{P}^2_{\bar{k}}$ with the normalization of the variety $X_{\bar{k}}$.

Proof. Diagram (4.1.2) clearly exists and commutes since both Z and X are regular varieties and hence reduced schemes. By Proposition 2.1.4, the morphism $f : Z \to X$ is flat and finite of degree 2.

We next make computations on the affine chart $U = (X_0 \neq 0)$, and by symmetry, analogous assertions are true over any chart of the form $(X_i \neq 0)$. Restricted to U, the top triangle of (4.1.2)

is dual to the following triangle of k-algebra morphisms:

(4.1.4)
$$\begin{tikzcd} & M := k[x_1,x_2,x_3]/(\alpha_0 + \sum \alpha_i x_i^2) \\ S := k[u_1,u_2,u_3]/(\alpha_0 + \sum \alpha_i u_i) \arrow[ur, "\bar{\mathbf{F}}^\sharp_{Z/k}"] \arrow[r, "\bar{g}^\sharp"] & R := \mathcal{O}_X|_U, \arrow[u, "f^\sharp"'] \end{tikzcd}$$

where $\bar{\mathbf{F}}^\sharp_{Z/k}$ is given by $u_i \mapsto x_i^2$. It is easy to check that $\bar{\mathbf{F}}_{Z/k}$ is flat and finite of rank 8 because M is a rank 8 free S-module with basis $\langle 1, x_1, x_2, x_3, x_1x_2, x_1x_3, x_2x_3, x_1x_2x_3\rangle$. Also, since f is flat and surjective, it is faithfully flat. Thus, $\bar{g}$ is flat, and hence finite of degree $8/2 = 4$.

To finish the proof, consider the geometric base change diagram (4.1.3). The morphism h, given explicitly as a morphism between the hyperplanes $Z^{\mathrm{red}}_{\bar{k}} \cong (\sum_{i=0}^3 \sqrt{\alpha_i} X_i = 0)$ and $(Z \times_{k,\mathbf{F}_k} \bar{k})^{\mathrm{red}} \cong (\sum_{i=0}^3 \alpha_i U_i = 0)$, is defined by the rule $[X_0 : X_1 : X_2 : X_3] \mapsto [X_0^2 : X_1^2 : X_2^2 : X_3^2]$. This is easily seen to be a finite dominant morphism of degree 4. The morphism $\bar{g}_{\bar{k}}$ is also a dominant morphism of degree 4 that factors h. This implies that $\bar{f}_{\bar{k}}$ is finite of degree 1, and hence a birational morphism. Since $Z^{\mathrm{red}}_{\bar{k}}$ is a hyperplane in $\mathbb{P}^3_{\bar{k}}$, it is isomorphic to $\mathbb{P}^2_{\bar{k}}$, and thus $\bar{f}_{\bar{k}}$ is a normalization morphism. ■

4.2. **Local ring of functions.** We compute $\mathcal{O}_X$ on an affine chart $(X_{i_0} \neq 0) \subseteq Z$, but for simplicity we assume $i_0 = 0$, as the computations on other charts are analogous by symmetry.

Proposition 4.2.1. *On the open $(X_0 \neq 0) \subseteq Z$, the ring of functions has presentation*

$$\mathcal{O}_X|_{(X_0 \neq 0)} = k[u_1,u_2,u_3,t_1,t_2,t_3]/(r_0,\dots,r_6),$$

with the relations r_i defined as:

$$\begin{array}{ll} r_0 := \alpha_0 + \alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3 & r_4 := t_2t_3 + u_1u_2u_3 + (u_1 + u_1^2)t_1 + u_1u_2t_2 + u_1u_3t_3 \\ r_1 := t_1^2 + u_2u_3 + u_2u_3^2 + u_2^2u_3 & r_5 := t_1t_3 + u_1u_2u_3 + u_1u_2t_1 + (u_2 + u_2^2)t_2 + u_2u_3t_3 \\ r_2 := t_2^2 + u_1u_3 + u_1^2u_3 + u_1u_3^2 & r_6 := t_1t_2 + u_1u_2u_3 + u_1u_3t_1 + u_2u_3t_2 + (u_3 + u_3^2)t_3. \\ r_3 := t_3^2 + u_1u_2 + u_1^2u_2 + u_1u_2^2 & \end{array}$$

Moreover, the inclusion of algebras $\mathcal{O}_X \subseteq \mathcal{O}_Z$ dual to the morphism $f : Z \to X$ is given by

$$k[u_1,u_2,u_3,t_1,t_2,t_3]/(r_0,\dots,r_6) \to k[x_1,x_2,x_3]/(\sum \alpha_i x_i^2),$$

via $u_i \mapsto x_i^2$ and $t_i \mapsto x_jx_k(1 + x_j + x_k)$, for each assignment of indices $\{i,j,k\} = \{1,2,3\}$.

Proof. Recall the diagram (4.1.4), and the notation established there. The S-algebra $R = \mathcal{O}_X|_{(X_0 \neq 0)}$ is flat and hence projective as an S-module. As S is isomorphic to a polynomial ring in two variables, over which all projective modules are free, R is actually a free S-submodule of rank 4 of the free S-module M of rank 8 with basis $\langle 1, x_1, x_2, x_3, x_1x_2, x_1x_3, x_2x_3, x_1x_2x_3\rangle$.

The derivation $\theta_{\mathbb{P}} := \Theta_{\mathbb{P}}|_{(X_0\neq 0)} = \sum_{i\neq 0}(x_i + x_i^2)\partial_{x_i}$ is S-linear because $S \subseteq R = \ker(\theta_{\mathbb{P}})$. Therefore, we may compute its matrix as an S-module morphism $M \overset{\theta_{\mathbb{P}}}{\to} M$:

$$\begin{array}{r c}
 & \begin{array}{cccccccc} 1 & x_1 & x_2 & x_3 & x_1x_2 & x_1x_3 & x_2x_3 & x_1x_2x_3 \end{array} \\
\begin{array}{r} 1 \\ x_1 \\ x_2 \\ x_3 \\ x_1x_2 \\ x_1x_3 \\ x_2x_3 \\ x_1x_2x_3 \end{array} &
\left(\begin{array}{c|ccc|ccc|c}
 & x_1^2 & x_2^2 & x_3^2 & & & & \\ \hline
 & 1 & & & x_2^2 & x_3^2 & & \\
 & & 1 & & x_1^2 & & x_3^2 & \\
 & & & 1 & & x_1^2 & x_2^2 & \\ \hline
 & & & & & & & x_3^2 \\
 & & & & & & & x_2^2 \\
 & & & & & & & x_1^2 \\ \hline
 & & & & & & & 1
\end{array}\right)
\end{array}$$

This is a block matrix, which makes computing its kernel easy:

$$\begin{pmatrix} 0 & A & 0 & 0 \\ 0 & 1 & B & 0 \\ 0 & 0 & 0 & C \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix} = \begin{pmatrix} Av_2 \\ v_2 + Bv_3 \\ Cv_4 \\ v_4 \end{pmatrix},$$

and this vector equals zero if and only if $v_4 = 0, v_2 = Bv_3$ and $ABv_3 = 0$. In our situation, the matrix $AB = 0$, and so we see that the kernel of M defined by $v_4 = 0, v_2 = Bv_3$. Thus, a basis of the kernel is given by the four elements

$$\begin{aligned}
\phi(t_0) :=& \quad 1 \\
\phi(t_1) :=& \quad x_2x_3 + x_2^2x_3 + x_2x_3^2, \\
\phi(t_2) :=& \quad x_1x_3 + x_1^2x_3 + x_1x_3^2, \\
\phi(t_3) :=& \quad x_1x_2 + x_1^2x_2 + x_1x_2^2,
\end{aligned}$$

and so $R = k[x_1^2, x_2^2, x_3^2, t_1, t_2, t_3] \subseteq k[x_1, x_2, x_3]/(\sum \alpha_i x_i^2)$.

Clearly, there is a surjective morphism ϕ from the polynomial algebra $k[u_1, u_2, u_3, t_1, t_2, t_3]$ onto R, defined by the rules $u_i \mapsto x_i^2$ and $t_i \mapsto \phi(t_i)$. The relations $r_0, \ldots, r_6$ listed above may be verified to be in $\ker\phi$ simply by writing the multiplication rules for the S-basis $\langle \phi(t_0), \phi(t_1), \phi(t_2), \phi(t_3)\rangle$. As a result, there is an induced surjective map of S-algebras,

$$\bar{\phi} : k[u_1, u_2, u_3, t_1, t_2, t_3]/(r_0, \ldots, r_6) \twoheadrightarrow R.$$

The domain is a free S-module with basis $\langle 1, t_1, t_2, t_3\rangle$, since all monomials in the t_i can be written as S-linear combinations of these elements modulo the relations r_i. Therefore, $\bar{\phi}$ is an isomorphism. ■

4.3. **An equation defining the singular locus.** We apply the Jacobian criterion to the presentation of $R = \mathcal{O}_X|_{(X_0\neq 0)}$ given in Proposition 4.2.1 to find the set of non-smooth points of X. It turns out that these points can be described set-theoretically as the vanishing locus of a single equation.

Proposition 4.3.1. *The non-smooth locus X^{sing} of X is set-theoretically equal to the codimension-1 locus defined by the single equation $\alpha_0 + \alpha_1 u_1^2 + \alpha_2 u_2^2 + \alpha_3 u_3^2 = 0$. In particular, X is not geometrically normal.*

Proof. The Jacobian matrix is as follows:

$$
\begin{array}{cc}
 & \begin{array}{cccccc} \partial_{u_1} & \partial_{u_2} & \partial_{u_3} & \partial_{t_1} & \partial_{t_2} & \partial_{t_3} \end{array} \\
\begin{array}{c} r_0 \\ r_1 \\ r_2 \\ r_3 \\ r_4 \\ r_5 \\ r_6 \end{array} &
\left(\begin{array}{ccc|ccc}
\alpha_1 & \alpha_2 & \alpha_3 & & & \\
 & u_3 + u_3^2 & u_2 + u_2^2 & & & \\
u_3 + u_3^2 & & u_1 + u_1^2 & & & \\
u_2 + u_2^2 & u_1 + u_1^2 & & & & \\ \hline
 & & & u_1 + u_1^2 & t_3 + u_1 u_2 & t_2 + u_1 u_3 \\
 & (*) & & t_3 + u_1 u_2 & u_2 + u_2^2 & t_1 + u_2 u_3 \\
 & & & t_2 + u_1 u_3 & t_1 + u_2 u_3 & u_3 + u_3^2
\end{array}\right)
\end{array}
$$

As R is a surface described in a 6-dimensional affine space, the singular locus is described by the ideal generated by the 4×4-minors of this matrix. As the Jacobian matrix comprises blocks in the form

$$\begin{pmatrix} A & 0 \\ * & B \end{pmatrix},$$

with $A = A^{3\times 4}$ and $B = B^{3\times 3}$, its 4×4 minors are either the product of two 2×2 minors of A and B, the product of an entry of A by the determinant of B, or the product of a 3×3 minor of A by an entry of B. Initially, the task of computing this ideal may appear daunting, but the following observation reduces the work dramatically.

Lemma 4.3.2. *Let B be the 3×3 matrix defined above.*

(1) The 2×2 minors of B are 0 in R.
(2) The diagonal entries of B generate the unit ideal in R.

Proof. Up to cyclic permutations of the indices $\{1, 2, 3\}$, there are only two types of 2×2-minors of B. A minor of the first type is $B_{3,3}$:

$$
\begin{aligned}
B_{3,3} &= (t_2 + u_1 u_3)(u_2 + u_2^2) + (t_3 + u_1 u_2)(t_1 + u_2 u_3) \\
&= u_1 u_2 u_3 + u_1 u_2^2 u_3 + (u_2 + u_2^2) t_2 + t_1 t_3 + u_2 u_3 t_3 + u_1 u_2 t_1 + u_1 u_2^2 u_3 \\
&= r_5 = 0.
\end{aligned}
$$

A minor of the second type is $B_{1,3}$:

$$
\begin{aligned}
B_{1,3} &= (t_1 + u_2 u_3)^2 + (u_3 + u_3^2)(u_2 + u_2^2) \\
&= t_1^2 + u_2^2 u_3^2 + u_2 u_3 + u_2^2 u_3 + u_2 u_3^2 + u_2^2 u_3^2 \\
&= r_1 = 0.
\end{aligned}
$$

This proves (1).

For (2), assume otherwise, and let $\mathfrak{m}$ be a maximal ideal containing the ideal generated by the entries of B. In the residue field $\kappa := R/\mathfrak{m}$, the image of the entry $u_i + u_i^2$ is 0, forcing $u_i = \varepsilon_i \in \kappa$, for $\varepsilon_i \in \{0, 1\}$. The relation $r_0 = 0$ implies $\alpha_0 + \varepsilon_1 \alpha_1 + \varepsilon_2 \alpha_2 + \varepsilon_3 \alpha_3 = 0$ in $k \subseteq \kappa$, which contradicts the algebraic independence of the α_i's. ■

From this lemma, it follows that the ideal generated by 4×4 minors of M is generated by the 3×3 minors of A. Denoting $h := \alpha_0 + \alpha_1 u_1^2 + \alpha_2 u_2^2 + \alpha_3 u_3^2$, these minors of A are $A_0 = 0$, $A_1 = (u_1 + u_1^2)h$, $A_2 = (u_2 + u_2^2)h$, $A_3 = (u_3 + u_3^2)h$. Lemma 4.3.2(2) shows these minors generate the principal ideal (h). ■

4.4. **The geometry of the singular locus.** Let C be the reduced subscheme corresponding to the non-smooth locus $X_{\bar{k}}^{\text{sing}} \subseteq X_{\bar{k}}$. By Proposition 4.3.1, the curve C is set-theoretically cut out by the equation $\alpha_0 + \alpha_1 u_1^2 + \alpha_2 u_2^2 + \alpha_3 u_3^2 = 0$, which is simply the square of the equation $\sqrt{h} = 0$ for

$$\sqrt{h} := \sqrt{\alpha_0} + \sqrt{\alpha_1} u_1 + \sqrt{\alpha_2} u_2 + \sqrt{\alpha_3} u_3.$$

We expect this equation to be insufficient to describe C scheme-theoretically, because X is not smooth along this locus, so the maximum ideal of the local ring $\mathcal{O}_{X,C}$ requires more than one generator. This is indeed the case, and the structure of C is as follows:

Proposition 4.4.1. *The curve C is isomorphic to a rational cuspidal curve with $h^1(C, \mathcal{O}_C) = 1$. The singular point of the curve sits below the point in $Z_{\bar{k}} \subseteq \mathbb{P}^3_{\bar{k}}$ described by the intersection of the three planes $(\sum \alpha_i^{1/2^j} X_i = 0)$, for $j = 1, 2, 3$.*

Proof of 4.4.1. Again we work over the chart $(X_0 \neq 0)$, and by symmetry our results will carry over to other opens $(X_i \neq 0)$. We must compute the quotient of the ring $R_{\bar{k}}/(\sqrt{h})$ by its nilradical ideal.

$$\bar{k}[u_1, u_2, u_3, t_1, t_2, t_3]/(\sqrt{h}, r_0, r_1, \dots, r_6).$$

The first two relations $r_0 = \alpha_0 + \sum_{i=1}^3 \alpha_i u_i$ and $\sqrt{h} = \sqrt{\alpha}_0 + \sum_{i=1}^3 \sqrt{\alpha_i} u_i$ are $\bar{k}$-linearly independent relations. Therefore, in the ring $R_{\bar{k}}/(\sqrt{h})$, we can solve for u_2 and u_3 in terms of u_1, and rewrite

$$R_{\bar{k}}/(\sqrt{h}) \cong \bar{k}[u, t_1, t_2, t_3]/(r_1, \dots, r_6),$$

where the variable u_1 is replaced by u and the variables u_2 and u_3 are replaced by the following expressions in u:

$$u_2(u) = \tfrac{(\alpha_0 + \sqrt{\alpha_0 \alpha_3}) + (\alpha_1 + \sqrt{\alpha_1 \alpha_3}) u}{\alpha_2 + \sqrt{\alpha_2 \alpha_3}}, \quad u_3(u) = \tfrac{\alpha_0 + \alpha_1 u + \alpha_2 u_2(u)}{\alpha_3}.$$

Hence the relations r_1, r_2, and r_3 read

$$\begin{aligned} r_1 = \quad & t_1^2 + c_{10} \quad + c_{11} u + c_{12} u^2 + c_{13} u^3 \\ r_2 = \quad & t_2^2 + 0 \quad\;\; + c_{21} u + c_{22} u^2 + c_{23} u^3 \\ r_3 = \quad & t_3^2 + 0 \quad\;\; + c_{31} u + c_{32} u^2 + c_{33} u^3, \end{aligned}$$

for explicitly determined coefficients $c_{ij} \in \bar{k}$ whose concrete description, for the sake of exposition, will be omitted but made available in an auxiliary file on the author's homepage. When written explicitly, it is straight-forward to check that these coefficients, for any pair $i, j \in \{1, 2, 3\}$, satisfy the following relation:

$$c_{i1} c_{j3} + c_{j1} c_{i3} = 0. \tag{4.4.2}$$

Make the following change of variables

$$\begin{aligned} s_1 &:= t_1 + \sqrt{c_{10}} + \sqrt{c_{12}} u \\ s_2 &:= t_2 + \sqrt{c_{22}} u \\ s_3 &:= t_3 + \sqrt{c_{32}} u, \end{aligned}$$

so that the relations r_1, r_2, r_3 become

$$\begin{aligned} r_1 = \quad & s_1^2 + c_{11} u + c_{13} u^3 \\ r_2 = \quad & s_2^2 + c_{21} u + c_{23} u^3 \\ r_3 = \quad & s_3^2 + c_{31} u + c_{33} u^3. \end{aligned}$$

The relations (4.4.2) imply $s_2^2 = \frac{c_{21}}{c_{11}} s_1^2$ and $s_3^2 = \frac{c_{31}}{c_{11}} s_1^2$, so the nilradical of $R_{\bar{k}}/(\sqrt{h})$ must contain the relations $r_2' := s_2 + \frac{\sqrt{c_{21}}}{\sqrt{c_{11}}} s_1$ and $r_3' := s_3 + \frac{\sqrt{c_{31}}}{\sqrt{c_{11}}} s_1$. By setting $s := s_1$, we obtain an isomorphism

$$R_{\bar{k}}/(\sqrt{h}, r_2', r_3') \cong \bar{k}[u, s]/(s^2 + u(c_{11} + c_{13}u^2), r_4, r_5, r_6).$$

Since $\bar{k}[u, s]/(s^2 + u(c_{11} + c_{13})u^2)$ is an integral domain of dimension 1, the relations r_4, r_5, r_6 are already 0 in this ring. Thus,

$$\mathcal{O}_C|_{(X_0 \neq 0)} = \bar{k}[u, \sqrt{u}(u + \sqrt{c_{11}/c_{13}})] \subseteq \bar{k}[\sqrt{u}].$$

From this description, is is clear that the only singular point of C is an ordinary cuspidal singularity of (wild) order 2 occurring at

$$X_1^2/X_0^2 = u = \sqrt{c_{11}/c_{13}}.$$

Moreover, one can verify that $\sqrt[4]{c_{11}/c_{13}} = \det(A_1)/\det(A_0)$ where the matrix A_1 is defined by replacing the first column of the following matrix A by the vector b:

$$A := \begin{pmatrix} \sqrt{\alpha_1} & \sqrt{\alpha_2} & \sqrt{\alpha_3} \\ \sqrt[4]{\alpha_1} & \sqrt[4]{\alpha_2} & \sqrt[4]{\alpha_3} \\ \sqrt[8]{\alpha_1} & \sqrt[8]{\alpha_2} & \sqrt[8]{\alpha_3} \end{pmatrix}, \qquad b := \begin{pmatrix} \sqrt{\alpha_0} \\ \sqrt[4]{\alpha_0} \\ \sqrt[8]{\alpha_0} \end{pmatrix}.$$

Cramer's rule, implies that the cusp of C sits below the intersection of the 3 planes

$$(\sum \sqrt[2^j]{\alpha_i} X_i = 0) \subseteq \mathbb{P}^3_{\bar{k}}, \text{ for } j = 1, 2, 3.$$

By symmetry, this is the only singular point of C. ■

5. Future research directions

Question 5.1. *Are there regular del Pezzo surfaces with positive irregularity in higher characteristic, that is, for $p \geq 3$?*

The inequality $q \geq \frac{d(p^2-1)}{6}$ of (0.2.2) relating the degree and irregularity becomes stronger as the characteristic grows, but it does not rule out the existence of such surfaces in any given characteristic. However, the author would find it surprising if examples exist in characteristic $p \geq 5$.

Question 5.2. *Are there regular del Pezzo surfaces X with positive irregularity over fields $k_X = H^0(X, \mathcal{O}_X)$ of inseparable degree $[k_X : k_X^p] \leq p^2$?*

As pointed out in Remark 3.2.3, the geometrically integral example X_1 may be constructed in characteristic 2 over a field of inseparable degree 2^3, and the geometrically non-reduced example was constructed over a field of inseparable degree 2^4. The case $[k_X : k_X^p] = p$ directly addresses a question of Kollár concerning 3-fold contractions [9, Rem. 1.2].

Question 5.3. *What is the geometry of the reduced structure on the geometric base change of the example X_2 constructed in §3.3?*

Presumably, one could explicitly compute local presentations of the ring of regular functions on $(X_2)_{\bar{k}}^{\mathrm{red}}$, as we did for the example X_1 in §4. This is left as an open exercise.

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